

Manifolds and Topology Preliminary Exam

August 30, 2007

No books, papers, or electronic devices may be used in this examination!

Write only your “codename” on your Blue Books, NOT your actual name!

This Examination has two Parts, I and II. Solutions of problems in separate Parts must be written in separate Blue Books.

There are four problems in each Part. Each problem has equal weight. The passing score will be based on your total score.

To receive full credit for a problem, the answer must be complete and correct. The scorers must not be expected to supply any missing parts of any answer.

Problems in each Part may be done in any order. You must clearly identify where the parts of your answers are. The scorers will not search at length for answers that are incomplete.

If you use a theorem, state it fully and concisely, or identify it clearly. In either case, verifying hypotheses explicitly is essential.

Do not make assumptions that trivialize a problem.

Manifolds/Topology Prelim

August 30, 2007

Please put the first 4 problems in one book and the second 4 problems in a second book. Please write your code name on both books.

1. a) Define the degree of a continuous map $f : S^n \rightarrow S^n$; where S^n is the n -sphere and $n > 0$. Write the degree as $\deg(f)$.

b) What is the degree of each of the following maps $f : S^n \rightarrow S^n$?

i) $f = c$, constant map with $c \in S^n$

ii) $f = id$, the identity map

iii) $f(x_0, \dots, x_n) = (x_1, x_0, x_2, \dots, x_n)$

iv) $f(x_0, \dots, x_n) = (-x_0, \dots, -x_n)$

c) What are the possible $\deg(f)$ of a map $f : S^n \rightarrow S^n$ such that $\deg(f \cdot f) = 1$?

d) Suppose $f : S^n \rightarrow S^n$ and $g = f \cdot f$. Suppose $\deg(g) < 50$. What are the possible values of $\deg(g)$?

2. State carefully the basic properties of a homology theory, i.e. the homotopy property, the exactness property and the excision property.

3. Let (X, x_0) denote a path connected space with base point. We say that $i : (A, a_0) \subseteq (X, x_0)$, an inclusion, is a retract, if there is a continuous map $f : X \rightarrow A$ so that $r \cdot i = id_A$, the identity on A . Y is another path connected space.

a) Let $(X, x_0) \times (Y, y_0)$ denote the cartesian product $X \times Y$ with base point (x_0, y_0) . Let $X \vee Y$ denote the subset $(p, q) \in X \times Y$ for which either $p = x_0$ or $q = y_0$, or both, (x_0, y_0) is the base point.

Show that $i : X \rightarrow X \times Y$ defined by $i(x) = (x, y_0)$ and $j : X \rightarrow X \vee Y$ defined by $j(x) = (x, y_0)$ are both retracts.

b) Prove that $\pi_1(X, x_0)$, the fundamental group, is always isomorphic to a subgroup of $\pi_1(X \times Y, (x_0, y_0))$ and a subgroup of $\pi_1(X \vee Y, (x_0, y_0))$.

c) Show that $S^1 \subseteq S^2$, the inclusion of the equator into the 2-dimensional sphere, is **NOT** a retract.

4. a) Let $p : \tilde{X} \rightarrow X$ be a covering space, $A \subseteq X$ a path connected and locally path connected subset of X . **Suppose** $p^{-1}(A)$ is path connected. Show that $p |_{p^{-1}(A)} : p^{-1}(A) \rightarrow A$ is a covering space.

b) Describe all compact X where $p : X \rightarrow S^1$ is a covering space.

5. Let M^n and N^k be smooth manifolds. Define what it means for a smooth $f : M^n \rightarrow N^k$ to be an immersion and what it means to be an embedding.
6. State Whitney's theorems about embeddings and immersions of a compact, smooth M^n in Euclidean space.
7. Let M^n be a compact smooth orientable manifold, $\omega \in D^{n-1}(M^n)$ a smooth $(n-1)$ differential form. Let $\tau = d\omega \in D^n(M^n)$. Show that $\int_{M^n} \tau = 0$.
8. Let T be a torus with k holes, $k > 0$, or equivalently let T be a 2-sphere with k handles attached. Let $f : T \rightarrow \mathbf{R}^3$ be an embedding, and put $M = f(T)$. M is a 2-dimensional submanifold of \mathbf{R}^3 .

Show that M has points of positive Gaussian curvature, as well as points of negative Gaussian curvature.