Manifolds and Topology Preliminary Exam
August 26, 2008

No books, papers or electronic devices may be used in this examination!

Write only your "codename" on your Blue Books, NOT your actual name!

This examination has two parts. Each problem has equal weight. The passing score will be based on your total score.

To receive full credit for a problem, the answer must be complete and correct. The scorers must not be expected to supply any missing parts of any answer.

Problems in each part may be done in any order. You must clearly identify where the parts of your answers are. The scorers will not search at length for answers that are incomplete.

If you use a theorem, state it fully and concisely, or identify it clearly. In either case, verifying hypotheses explicitly is essential.

Do not make assumptions that trivialize a problem.

Part 1.

A1 A continuous map $i : A \to X$ is a retraction, if there is a continuous $\rho : X \to A$ so that $\rho \cdot i = Id_A$. Let

$$B^n = \{\vec{x} \in \mathbf{R}^n \mid ||\vec{x}|| \leqq 1\}, \quad n > 1$$
$$S^{n-1} = \{\vec{x} \in \mathbf{R}^n \mid ||\vec{x}|| = 1\}.$$

i) Show that the inclusion $S^{n-1} \subseteq B^n$ is **not** a retraction.

ii) Show that $S^{n-1} \subseteq \{\vec{x} \in \mathbf{R}^n \mid ||\vec{x}|| > 0\}$ is a retraction.

iii) Let $x_1, \ldots, x_k$ be distinct points in $\mathbf{R}^n$, $n > 1$ $k \geq 1$. Show that $\mathbf{R}^n - \{x_1, \ldots, x_k\} \subseteq \mathbf{R}^n$ is **not** a retraction.

A2 (i) Calculate the degree of $f : S^n \to S^n$ defined by $f(x_0, \ldots, x_n) = (-x_0, \ldots, -x_n)$.

(ii) Suppose f is continuous map, $f : S^n \to S^n$, with distance $(\vec{x}, f(\vec{x})) < 1$, all x. (Recall that the radius of S^n is 1). Show degree $f = 1$.

(iii) What are the possible degrees of $f : S^n \to S^n$ when (degree $(f))^2$ = degree (f)? Give examples of such maps.

A3 Let X be path connected and locally path connected, $x_0 \in X$. State carefully the theorem which classifies the covering spaces $p : \tilde{X} \to X$ with given $\tilde{x}_0 \in \tilde{X}$ so that $f(\tilde{x}_0) = x_0$.

A4 Describe all covering spaces of $\mathbf{R}P^2 \times \mathbf{R}P^2$, where $\mathbf{R}P^2$ is the real projective plane.

Part 2

B1 Define a diffeomorphism $f : M^n \to N^n$, where $M^n \in N^n$ are n-dimensional manifolds; f is a diffeomorphism in the following parts.

(i) If $0 \subseteq M^n$ is open and non-empty, show that $f : 0 \to f(0)$ is also a diffeomorphism.

(ii) Show that $g : M^n \times M^n \to M^n \times N^n$ is a diffeomorphism when $g(x, y) = (y, f(x))$.

(iii) Let $h : \mathbf{R}^1 \to \mathbf{R}^1$ be given by $h(x) = x^n$, $n = 2k + 1$, $k > 0$. Show that h is 1-to-1 but **not** a diffeomorphism.

B2 Let $\omega = (\cos x + e^x yz)dx + e^x zdy + e^x ydz$ be a differential 1-form on $\mathbf{R}^3$. Calculate $d\omega$. If $d\omega = 0$, find τ so that $d\tau = \omega$. (τ is smooth function).

B3 Show that the space of unit tangent vectors on $S^2 \subseteq \mathbf{R}^3$ is homeomorphic (or diffeomorphic) to the space of oriented triples $\{\vec{u}, \vec{v}, \vec{w}\}$ where $\vec{u}, \vec{v}$ and $\vec{w}$ are mutually orthogonal unit vectors in $\mathbf{R}^3$. Identify this as a group of matrices.

B4 Define immersions and embeddings $M^n \to N^k$, $n \leq k$. Show that there is an immersion $f : S^1 \to \mathbf{R}^2$ which cannot be approximated to arbitrary small accuracy, by an embedding; yet if $j : \mathbf{R}^2 \to \mathbf{R}^3$, $j(x, y) = (x, y, 0)$, then $j \cdot f$ can be so approximated by an imbedding.