Manifolds/Topology Prelim
April 20, 2009

This exam has two parts. A1,A2,A3, and A4 and B1,B2,B3 and B4. Please use separate blue books for each part A and B. Be sure to put your code name on each book and indicate clearly which problems are in each book. **Do not** write your real name on any book. Please explain your work clearly and indicate clearly what results you are using in your explanation.

A1 (a) What are the fundamental groups of the circle S^1 and the real projective plane RP^2?

(b) Show that there are no spaces X, having the homotopy-type of S^1 and Z having the homotopy-type of RP^2, with a covering space $Z \to X$.

A2.(a) If $f : S^n \to S^n$ is a continuous map of a unit sphere to itself, $n > 0$, define the degree deg(f).

(b) If n is odd, $n > 1$, and

$$f(x_0, \ldots, x_n) = (-x_0, x_1, -x_2, x_3, \ldots).$$

What is deg(f)?

(c) Suppose $n > 3$ is even, and

$$g(x_0, \ldots, x_n) = (-x_0, \ldots, -x_n)$$
$$h(x_0, \ldots, x_n) = (x_1, x_0, x_3, x_2, x_4, x_5, x_6, \ldots x_{n-1}, x_n)$$

are g and h homotopic? Explain.

A3. (a) Find a continuous map $f : T \to T$, where $T = S^1 \times S^1$, which is homotopic to the identity, but f has no fixed point.

(b) Let $X = S^1 \vee S^1$, the 1 point union or wedge. Show that any continuous $g : X \to X$ which is homotopic to the identity has a fixed point. (Hint: you may use the Lefschetz fixed-point theorem)

A4. (a) State the exactness property for a pair of spaces (X, A) in singular homology.

(b) If $H_i(A) \to H_i(X)$, $i > 0$, is 1-to-1, then show that $H_{i+1}(X) \to H_{i+1}(X, A)$ is onto.

B1. State Whitney's theorems about the immersion and embeddings of a compact, smooth n-manifold in Euclidean space.

B2. For each of the following surfaces, identify the points of positive and of negative curvature (Gaussian curvature).

(a) The torus (rotate $(x-2)^2 + z^2 = 1$ around the z-axis)

(b) The surface obtained by rotating $z = y^2 + 2$ around the y-axis.

(c) The surface obtained by rotating $z = \frac{1}{y^2+1}$ around the y-axis.

B3. (a) Describe an immersion of a non-compact manifold M^n in a Euclidean space, which is 1-to-1, but not an embedding.

(b) Show that there is no immersion of S^1 in $\mathbf{R}^1$.

B4. Let $M^3 = \mathbf{R}^3 - \{\text{z} - \text{axis}\}$. Consider the following two vector fields

$V =$ all unit vectors parallel to the z-axis pointing in the negative direction.

$W = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

If $P \in M^3$, is there a 2-dim submanifold of M^3, in a neighborhood of P, for which V and W span the tangent space? Explain!