Real Analysis Preliminary Exam

August 27, 2008

Write your **codename***, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.*

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems silly. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

All problems are worth 10 points. The parts of problems have equal weights.

1. Let $f:[0,1]\to\mathbb{R}$ be absolutely continuous, and assume that $f'\in L^2([0,1])$ and that $f(0)=0$. Show that the following limit exists, and compute its value.

$$\lim_{x\to 0+} x^{-1/2} f(x)$$

2. Let K be a compact subset of $\mathbb{R}$, and suppose that $f:K\to\mathbb{R}$ and $f_n:K\to\mathbb{R}$, $n=1,2,\ldots$, are continuous. Suppose that, for every $x\in K$, $f_{n+1}(x)\le f_n(x)$, $n=1,2,\ldots$, and $\lim_{n\to\infty} f_n(x)=f(x)$.

a. Show that $f_n(x)\to f(x)$ uniformly on K as $n\to\infty$.

b. Give an example to show that the compactness of K is necessary.

3. Recall that a topological space is *locally compact* if every point has a compact neighborhood. Prove or disprove:

a. Every locally compact metric space is complete.

b. Every locally compact inner product space is finite dimensional.

4. Assuming that $f\in L^3([0,1])$, show that the following integral is finite.

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx$$

5. Assume that S is a Lebesgue measurable subset of $\mathbb{R}$, with $\mu(S) < \infty$. Show that

$$\mu\left(S \cap [R, \infty)\right) \to 0 \text{ as } R \to \infty.$$

6. Assume that S is a Lebesgue measurable subset of $[0, 2\pi]$. Show that

$$\lim_{n\to\infty} \int_S \sin nx \, dx = 0.$$

7. Assume that S is a Lebesgue measurable subset of $[0,1]$ and that S has positive measure. Show the existence of two points $x \in S$ and $y \in S$ such that $x - y$ is rational.

8. Suppose that $f \in L^1(\mathbb{R})$ and that $f(x) > 0$, for all $x \in \mathbb{R}$. Let $\hat{f}$ be the Fourier transform of f. Show that $\left|\hat{f}(t)\right| \leq \hat{f}(0)$ for all $t \in \mathbb{R}$, with equality if and only if $t = 0$.