Real Analysis Preliminary Exam

April 23, 2009

Write your **codename**, *not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.*

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

All problems have equal weight.

1. Let $E \subset \mathbb{R}$ have finite Lebesgue measure, and let χ_E be the characteristic function of E. Show that the following limit exists, and compute its value.

$$\lim_{x\to\infty} \int_x^\infty \frac{\chi_E(t)}{t}\,dt$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on X. Show that $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

3. Suppose that $F:[a,b]\to\mathbb{R}$ and $G:[a,b]\to\mathbb{R}$ are absolutely continuous. Show that FG is absolutely continuous.

4. Recall that, if (X,ρ) and (Y,σ) are metric spaces, a function $f : X \to Y$ is called Lipschitz continuous if there exists a constant $\lambda \ge 0$ such that

$$\sigma\big(f(x), f(y)\big) \le \lambda\rho(x,y) \quad \text{for all} \quad x, y \in X .$$

Let X be a compact metric space, and let $Y = \mathbb{R}$, Show that the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm.

5. Let $f:[0,1]\to[0,1]$ be a nondecreasing continuous function satisfying $f(0)=0$ and $f(1)=1$. Prove or give a counterexample: $\int_0^1 f'(x)\,dx = 1$.

6. Assuming that $f \in L^2([0,1])$ and that $f(x) \geq 0$ for $x \in [0,1]$, show that the following integral is finite.

$$\int_0^1 \sqrt{\frac{f(x)}{x}}\,dx$$

7. Using an appropriate Fourier series, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

8. Let $f \in L^1(\mathbb{R})$ and suppose that $f(t) > 0$ for all $t \in \mathbb{R}$. Let $\alpha > 0$, let $g(t) = e^{i2\pi\alpha t} f(t)$, and let $\hat{g}$ be the Fourier transform of g. Compute $\sup_{x \in \mathbb{R}} |\hat{g}(x)|$, and show that the supremum is achieved at exactly one point.