Placement Test 1 (Algebra)

You may cite standard theorems below, without proof. These include the Sylow theorems, the structure theorem for finitely generated modules over a principal ideal domain.

1. (a) Define "group".

(b) Define "Abelian group".

(c) Which finite non-Abelian group has the fewest elements?

2. (a) Define "field".

(b) Define "subfield".

(c) How many fields are there with exactly 8 elements? Explain your answer.

(d) How many fields are there with exactly 26 elements? Explain your answer.

(e) Let F be a subfields of a finite field E. Define the Galois group G of E over F. Define the degree $[E : F]$ of E over F. Is it true that the order of G must be $[E : F]$? Explain your answer.

3. (a) Explain how to use the Euclidean algorithm to compute the gcd of 24885 and 45780. Compute the first three steps.

(b) Define "ring". (As I use the term, a ring need not be commutative, but should have multiplicative identity.)

(c) Define "subring".

(d) Define "integral domain".

(e) Define "ideal".

(f) Define "principal ideal domain".

(e) Let R be an integral domain. Does there necessarily exist a field F such that R is a subring of F? Explain your answer briefly.

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4. (a) Define "vector space over the real numbers".

(b) Define "linear transformation" from one vector space over the real numbers to another.

(c) Let V be a vector space over the real numbers, and let $T : V \to V$ be a linear transformation. Define the characteristic polynomial of T.

(d) Let $n \geq 1$ be an integer. Let M be a $n \times n$ matrix with real entries. Let M^t be the transpose of M. Show that there is a matrix X with real entries such that $X^2 = M^t M$. (Hint: You may assume the "Spectral Theorem" which states that any symmetric matrix with real entries is orthogonally diagonalizable over the reals.)

5. Let R be a ring.

(a) Define a "module" over R.

(b) Define a "cyclic module" over R.

(c) Define "finitely generated module" over R.

6. Give an example of a ring R and a finitely generated module M over R such that M is not isomorphic to a direct sum of cyclic modules over R.

7. Let $R := \mathbb{Z}/10\mathbb{Z}$. How many cyclic modules are there over R? Describe all of them.