Placement Test 3 (Analysis)

You may cite standard theorems below, without proof.

1. Let $\{a_k\}$ be a sequence of real numbers.

(a) Define what it means to say that "a_k is convergent".

(b) Define what it means to say that "a_k is Cauchy".

(c) Assume that $\{a_k\}$ is convergent. Show that $\{a_k\}$ is Cauchy.

2. Show that there exists an *uncountable* collection $\mathcal{S}$ of subsets of $\mathbb{Z}$ such that, for for all $S, T \in \mathcal{S}$ we have: either $S \subseteq T$ or $T \subseteq S$. (*Hint:* First, consider this problem with $\mathbb{Z}$ replaced by $\mathbb{Q}$.)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Let $x \in \mathbb{R}$. Define "f is continuous at x".

(b) Let $S \subseteq \mathbb{R}$. Define "f is uniformly continuous on S".

(c) Let $x \in \mathbb{R}$. Define "f is differentiable at x".

(d) Assume, for all $x \in [0, 1]$, that f is continuous at x. Show that f is uniformly continuous on $[0, 1]$.

4. Let $f_1, f_2, \ldots : \mathbb{R} \to \mathbb{R}$ be a sequence of functions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define what it means to say that $\{f_k\}$ is "uniformly convergent" to f.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume that $\{f_k\}$ is uniformly convergent to f. Assume, for all integers $k \geq 1$, that f_k is continuous at 0. Show that f is continuous at 0.

(c) Assume, for all integers $k \geq 1$, that f_k is differentiable at all real numbers. Assume, for all integers $k \geq 1$ and all $x \in \mathbb{R}$, that $|f_k'(x)| \leq 1$. Assume, for all integers $k \geq 1$, that $f_k(0) = 0$. Show that there is an increasing sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that, for every $x \in \mathbb{Q}$, we have that $\{f_{n_k}(x)\}$ is a convergent sequence of real numbers. (Hint: First show, for all $x \in \mathbb{R}$, that $\{f_k(x)\}$ is bounded. Then use Cantor diagonalization.)

5. Let (X, d) be a compact metric space and let $\mathcal{U}$ be an open cover of X. For all $x \in X$, for all $\delta > 0$, let

$$B(x, \delta) := \{y \in X \mid d(x, y) < \delta\}.$$

Show that there exists $\delta > 0$ such that for all $x \in X$, there exists $U \in \mathcal{U}$ such that $B(x, \delta) \subseteq U$.