A fragmentation-coagulation chain, random walk on permutations, and Schramm's coupling.

Ofer Zeitouni

September, 19

Abstract

Consider a Markov chain with state space $\Sigma_1 = \{x_1 \geq x_2 \geq ... \geq 0 : \sum x_i = 1\}$. At each step, two parts x_i are chosen independently with replacement, with probability of chosing part i being x_i. If the same part was chosen, it splits (uniformly). If two different parts were chosen, they merge. A discrete approximation of this chain (with $\sum x_i = N$ and x_i non-negative integers) is given by the cycle structure of permutations performing a random walk by random transpositions.

Diaconis and Shashahani showed that the transposition walk exhibits a threshold phenomenum: starting from the identity, at time $(1/2)n \log n + Kn$, the variation distance d_K between the law of the walk and the uniform law is uniformly (in n) bounded below, while $d_K \to 0$ as K increases. Their proof involves the representation theory of the symmetric group. On the other hand, Aldous conjectured that at time larger than $N/2$, the law of the "large parts" converges to the uniform law, weakly.

Motivated by the study of "infinite permutations", Vershik observed that the invariant measure of the coagulation-fragmentation chain is the Poisson-Dirichlet measure, and conjectured that it is unique. This was proved in 2003 to be the case by Diaconis, Mayer-Wolf, Zerner and myself, using representation theoric tools and the relation with the discrete chain.

In 2004, Schramm proved Aldous' conjecture by using a clever coupling, bypassing the representation theoretic analysis. Along the way, he also provided a new proof of Vershik's conjecture. I will explain Schramm's ideas. If time permits, I will explain how a variant of his coupling (also devised by him) gives a proof of the threshold phenomenum. The interest in this is that Schramm's ideas extend beyond random transpositions to situations in which the representation theoretic analysis becomes challenging. This part is work in progress with Nathanael Berestycki and Oded Schramm.