FM 5001 Fall 2007, Final Exam
Handout date: Wednesday 12 December 2007

1. Definitions: Complete the following sentences.

a. (5 pts.) Two quadratic forms Q and Q' are said to be **equivalent** if ...

b. (5 pts.) The **image** of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^k$ is ...

c. (5 pts.) A function $f : X \to Y$ is **onto** if ...

d. (5 pts.) Let Q be a positive semidefinite quadratic form, and let B be its polarization. Then the Cauchy-Schwarz inequality states that ...

e. (5 pts.) The **dimension** of a subspace S of $\mathbb{R}^n$ is ...

f. (5 pts.) A matrix $K \in \mathbb{R}^{n\times n}$ is **orthogonal** if ...

2. True or False. (No partial credit.)

a. (5 pts.) If a square matrix has a left inverse, then it has a right inverse, and these two inverses are equal.

b. (5 pts.) If $f(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ is the characteristic polynomial of a matrix M, then $c_nM^n + c_{n-1}M^{n-1} + \cdots + c_1M + c_0I = 0$.

c. (5 pts.) If a square matrix with integer entries has nonzero determinant, then its inverse has integer entries.

d. (5 pts.) If a linear transformation $f : V \to W$ is onto, then it is invertible, even though the dimensions of V and W may not be equal.

e. (5 pts.) Let $A, B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear forms (not necessarily symmetric). Assume, for all $v \in \mathbb{R}^n$, that $A(v, v) = B(v, v)$. Then $A = B$.

f. (5 pts.) For any $M \in \mathbb{R}^{n\times n}$, there exists an invertible matrix $L \in \mathbb{R}^{n\times n}$ such that LM is diagonal.

g. (5 pts.) If you perform an elementary row or column operation to a matrix with nonzero determinant, then the result also has nonzero determinant.

h. (5 pts.) Every symmetric matrix is rotationally diagonalizable.

3. Computations. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have exactly four elements in them? (You need not list them, just give the count.)

b. (5 pts.) Let

$$C := \begin{bmatrix} 2 & 7 & 0 & 3 \\ 3 & -7 & 2 & 7 \\ -1 & -2 & 6 & -3 \end{bmatrix}.$$

Using elementary row and column operations, put C in fully canonical form.

c. (5 pts.) Find the dimensions of the kernel and image of $L_C : \mathbb{R}^4 \to \mathbb{R}^3$, where

$$C := \begin{bmatrix} 2 & 7 & 0 & 3 \\ 3 & -7 & 2 & 7 \\ -1 & -2 & 6 & -3 \end{bmatrix}.$$

d. (5 pts.) Let

$$M := \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} \end{bmatrix}.$$

Find a rotation matrix K such that the rows of KM are orthogonal to one another.

e. (5 pts.) Let K and M be the matrices in 3.d. Note that $(1, -\sqrt{2}, 1)$ is orthogonal to the rows of M. Find an orthogonal 3×3 matrix L such that KML is a 2×3 diagonal matrix.

f. (5 pts.) Find the inverse of

$$D := \begin{bmatrix} 3 & 2 & 0 & 0 \\ 7 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix}.$$

g. (5 pts.) Give an example of a 3×3 matrix A that is not diagonalizable.

h. (5 pts.) Let A be the matrix you chose in 3.g. Find the eigenvalues of A. For each eigenvalue of A, find the dimension of the corresponding eigenspace.

i. (5 pts.) Let

$$D := \begin{bmatrix} 3 & 2 & 0 & 0 \\ 7 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

be as in 3.f. Determine whether D is diagonalizable over the *real* numbers. (You don't need to diagonalize D, but you must give a clear reason why D is or is not diagonalizable over the real numbers.)

j. (5 pts.) Give an example of two piecewise constant random variables X and Y that are independent.

k. (5 pts.) Let X and Y the piecewise constant random variables you chose in 3.j. Write down the partition P of X and the partition Q of Y. Compute $\mathrm{E}[X|P]$ and $\mathrm{E}[X|Q]$.