FM 5002 Spring 2008, Final Exam
Handout date: Wednesday 7 May 2008

**Name (Print):** **Student ID or email address:**

1. Definitions: Complete the following sentences.

a. (5 pts.) The **variance** of a PCRV $X$ is $\mathrm{Var}[X] := \cdots$.

b. (5 pts.) The **covariance** of two PCRVs $X$ and $Y$ is $\mathrm{Covar}[X, Y] := \cdots$.

c. (5 pts.) Two PCRVs $X$ and $Y$ are **independent** if ...

d. (5 pts.) A sequence $X_1, X_2, \ldots$ of PCRVs is a **PCRV approximation** if ...

e. (5 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. The **third-order Maclaurin approximation** to $f$ is ...

f. (5 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. The **3-jet** at 0 of $f$ is ...

g. (5 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is said to be **strictly decreasing** on an interval $I$ if, for all $a, b \in I$, with $a < b$, we have: ...

h. (5 pts.) The **gradient** $\nabla f$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$(\nabla f)(x, y) = \cdots$$

i. (5 pts.) The **Hessian** $Hf$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$(Hf)(x, y) = \cdots$$

j. (5 pts.) A **vector field** on $\mathbb{R}^2$ is ...

k. (5 pts.) A **flowline** for a vector field $V$ on $\mathbb{R}^2$ is a function $c : (a, b) \to \mathbb{R}^2$ such that, for all $t \in (a, b)$, we have $(d/dt)(c(t)) = \cdots$

2. True or False. (No partial credit.)

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a. (5 pts.) If two functions have the same gradient, then they must be equal.

True False

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b. (5 pts.) Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be any two smooth functions. Let $i := \sqrt{-1}$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x + iy) = [u(x, y)] + i[v(x, y)]$. Then $f$ is complex analytic.

True False

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c. (5 pts.) Let $V$ be a vector field on $\mathbb{R}^2$. Then there is a flowline $c : (-\infty, \infty) \to \mathbb{R}^2$ for $V$, footed at $(0, 0)$, which is defined on the entire real line $\mathbb{R} = (-\infty, \infty)$.

True False

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d. (5 pts.) If $\phi : \mathbb{R} \to \mathbb{R}^2$ and $\psi : \mathbb{R} \to \mathbb{R}^2$ are both flowlines for a vector field $V$ on $\mathbb{R}^2$, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \mathrm{dist}(\phi(t), \psi(t))$ is constant. Here, the distance between two points $(a, b)$ and $(p, q)$ in the plane $\mathbb{R}^2$ is defined by the formula

$$\mathrm{dist}((a, b), (p, q)) = \sqrt{(p - a)^2 + (q - b)^2}.$$

True False

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e. (5 pts.) If $Q : \mathbb{R}^n \to \mathbb{R}$ is a positive semidefinite quadratic form, and if $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the polarization of $Q$, then, for all $v, w \in \mathbb{R}^n$, we have: $[B(v, w)]^2 \leq [Q(v)][Q(w)]$.

True False

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f. (5 pts.) Any positive semidefinite symmetric matrix is the variance-covariance matrix of a collection of PCRVs.

True False

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g. (5 pts.) If $f : \mathbb{R} \to \mathbb{R}$ is smooth and one-to-one, then, for all $x \in \mathbb{R}$, we have: $f'(x) \neq 0$.

True False

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h. (5 pts.) If $f : \mathbb{R} \to \mathbb{R}$ is constant, then, for all $x \in \mathbb{R}$, we have: $f'(x) = 0$.

True False

Grading Page

**Not for student use. Please leave blank for grader. Thanks.**

1.

2.

3a.

3b.

3c.

3d.

3e.

3f.

3g.

3h.

3i.

3j.

3k.

3. Computations. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Let $X$ be a PCRV with $\Pr[X = 2] = 0.7$ and $\Pr[X = 6] = 0.3$. Compute $\mathrm{E}[X]$.

b. (5 pts.) Let $X$ be a PCRV with $\Pr[X = 2] = 0.7$ and $\Pr[X = 6] = 0.3$. Compute $\mathrm{Var}[X]$.

c. (5 pts.) Let $X$ and $Y$ be a PCRVs with $\mathrm{SD}[X] = 5$, $\mathrm{SD}[Y] = 2$ and $\mathrm{Corr}[X, Y] = 0.8$. Compute $\mathrm{Var}[X + Y]$.

d. (5 pts.) Let $X$ and $Y$ be independent standard PCRVs. Find an upper triangular matrix $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$ such that, if we set

$$\begin{array}{rclcl} S & := & aX & + & bY \\ T & := & & & cY, \end{array}$$

then the variance-covariance matrix of $S$ and $T$ is $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

e. (5 pts.) Compute $\int_{-\infty}^{\infty} ((e^{4x+3} - e^7)_+ + x^{101})e^{-x^2/2}\, dx$.

f. (5 pts.) Let $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ be the standard Brownian motion approximation. Compute $\lim_{N\to\infty} \mathrm{E}[(e^{4X_1^{(N)}+3} - e^7)_+ + (X_1^{(N)})^{101}]$.

g. (5 pts.) Let $F$ be the expression of $x$, $y$ and $z$ given by $F = e^{xy}[\cos(yz)]$. Compute $dF$, the exterior derivative of the 0-form $F$.

h. (5 pts.) Find the gradient of $f$, where $f(x, y, z) = e^{xy}[\cos(yz)]$.

i. (5 pts.) Let $\omega$ be the one-form in the single variable $x$ defined by

$$\omega = [(2x^2 + e^{\cos x})^{100} - 2e^x(\sec^2(x^5)) + \tan^4((x^7)(\ln x))]\, dx.$$

Compute $d\omega$.

j. (5 pts.) Find the Hessian of $f$, where $f(x, y) = 3x^2 + 5xy^2$.

k. (5 pts.) Find the maximum of $16x - 2y$ subject to the constraint $x^4 + y^4 = 1$. That is, find the maximum element of $\{ \quad 16x - 2y \quad | \quad x, y \in \mathbb{R}, \quad x^4 + y^4 = 1 \quad \}$ .