FM 5002 Spring 2008, Midterm #1
Handout date: Wednesday 05 March 2008

1. Definitions: Complete the following sentences.

a. (5 pts.) Let Y be a PCRV. Then the **variance**, $\mathrm{Var}[Y]$, of Y is defined by $\mathrm{Var}[Y] = \cdots$

b. (5 pts.) Let X and Y be PCRVs. Then the **covariance**, $\mathrm{Covar}[X, Y]$, is defined by $\mathrm{Covar}[X, Y] = \cdots$

c. (5 pts.) Let A and B be events. Then the **conditional probability**, $\Pr[A|B]$, of A given B is defined by $\Pr[A|B] = \cdots$

d. (5 pts.) Let X be a PCRV and let A be an event. Then the **conditional expectation**, $\mathrm{E}[X|A]$, of X given A is defined by $\mathrm{E}[X|A] = \cdots$

e. (5 pts.) Let X and Y be PCRVs. Then the **conditional expectation**, $\mathrm{E}[X|Y]$, of X given Y is defined by $(\mathrm{E}[X|Y])(\omega) = \cdots$.

f. (5 pts.) The odds form of Bayes' formula reads

$$\mathrm{Odds}[A|B] = (\mathrm{Odds}[A])(\mathrm{LQ}^A[B]),$$

where $\mathrm{LQ}^A[B] := \cdots$.

2. True or False. (No partial credit.)

a. (5 pts.) For any matrix M, there exists a matrix A such that $M = AA^t$.

b. (5 pts.) If two PCRVs have the same distribution, then they must be equal.

c. (5 pts.) Let X and Y be independent PCRVs. Then $\mathrm{E}[X|Y]$ must be constant.

d. (5 pts.) Let X and Y be PCRVs, let $\mathcal{P}$ be the partition of Y and assume that X is $\mathcal{P}$-measurable. Then $\mathrm{E}[X|Y]$ must be constant.

e. (5 pts.) Let X be a PCRV and let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of Ω by fUofIs (finite unions of intervals). Assume that $\mathcal{Q}$ is coarser than $\mathcal{P}$. Then

$$\mathrm{E}[\ \mathrm{E}[X|\mathcal{P}]\ |\ \mathcal{Q}\ ] = \mathrm{E}[\ X\ |\ \mathcal{Q}\ ].$$

3. Computations. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Let X be the PCRV defined by

$$X(\omega) = \begin{cases} 1, & \text{if } 0 \leq \omega < 0.3 \\ 0, & \text{if } 0.3 \leq \omega \leq 1 \end{cases}$$

Compute $\mathrm{E}[X]$ and $\mathrm{SD}[X]$.

b. (5 pts.) Let X be as in 3a and let Y be the PCRV defined by

$$Y(\omega) = \begin{cases} 1, & \text{if } 0 \leq \omega < 0.5 \\ -1, & \text{if } 0.5 \leq \omega \leq 1 \end{cases}$$

Compute $\mathrm{E}[XY]$ and $\mathrm{Covar}[X, Y]$.

c. (5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} e^{4x} e^{-x^2/2}\, dx$.

d. (5 pts.) Compute $\displaystyle\int_{1}^{\infty} e^{6x} e^{-x^2/2}\, dx$.
(NOTE: The lower limit of integration is 1, not $-\infty$.)

e. (5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} x^9 e^{-x^2/2}\, dx$.

f. (5 pts.) Let

$$M := \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}.$$

Find an upper triangular matrix A such that $AA^t = M$.

g. (5 pts.) Let M be as in 3f. Find a lower triangular matrix B such that $B^t B = M$.

h. (5 pts.) Let Z_1 and Z_2 be independent standard PCRVs. Find real numbers a_1, a_2, b_1, b_2 such that

$$\mathrm{Var}[\quad a_1 Z_1 + a_2 Z_2 \quad] \quad = \quad 4, \qquad\qquad \mathrm{Var}[\quad b_1 Z_1 + b_2 Z_2 \quad] \quad = \quad 9,$$

and

$$\mathrm{Covar}[\quad a_1 Z_1 + a_2 Z_2 \quad , \quad b_1 Z_1 + b_2 Z_2 \quad] \quad = \quad 3.$$

i. (5 pts.) Let P and Q be PCRVs. Let $s := \mathrm{SD}[P]$, let $t := \mathrm{SD}[Q]$ and let $\rho := \mathrm{Corr}[P, Q]$. For all x, let $f(x) := \mathrm{SD}[P - xQ]$. Find the value of x such that $f(x)$ is minimized. (Of course, x will depend on s, t and ρ.)