

FM 5002 Spring 2008, Midterm #2
Handout date: Wednesday 09 April 2008

1. Definitions: Complete the following sentences.

- a. (5 pts.) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. The two-jet of f at 0 is the following ordered triple of numbers: ...
- b. (5 pts.) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. The second-order Maclaurin approximation for f is the second degree polynomial p such that ...
- c. (5 pts.) Let X be a PCRV, and let \mathcal{P} be a partition of Ω by finite unions of intervals. Then $E[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in P \in \mathcal{P}$ (and if P is not of zero size), then $(E[X|\mathcal{P}])(\omega) = \dots$
- d. (5 pts.) Let X be a PCRV. Then the **variance** of X is defined by $\text{Var}[X] = \dots$.

2. True or False. (No partial credit.)

- a. (5 pts.) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be smooth functions. Suppose that $f(0) = g(0) = 1$, that $f'(0) = g'(0) = 0$ and that $f''(0) = g''(0)$. Then $\lim_{N \rightarrow \infty} [f(x/\sqrt{N})]^N = \lim_{N \rightarrow \infty} [g(x/\sqrt{N})]^N$.
- b. (5 pts.) If X and Y are any two independent PCRVs, then $E[XY] = E[X]E[Y]$.
- c. (5 pts.) Let \mathcal{P} and \mathcal{Q} be partitions of $\Omega = [0, 1]$ by finite unions of intervals. Assume that \mathcal{P} is finer than \mathcal{Q} . Then any \mathcal{P} -measurable PCRV is \mathcal{Q} -measurable.
- d. (5 pts.) Let X and Y be PCRVs. Assume that X and Y have the same distribution. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then $E[\phi(X)] = E[\phi(Y)]$.
- e. (5 pts.) Let f be differentiable, and assume that f' is continuous. If $f'(x) \neq 0$, for all $x \in \mathbb{R}$, then f is one-to-one.
- f. (5 pts.) Let f be differentiable, and assume that f' is continuous. If f is one-to-one, then $f'(x) \neq 0$, for all $x \in \mathbb{R}$.

3. Computations. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Let $W_t^{(N)}$ be the standard approximation to Brownian motion. Compute $\lim_{N \rightarrow \infty} \mathbb{E}[(W_6^{(N)})^2]$. Don't just set this up as an integral; compute the integral.

b. (5 pts.) Let $W_t^{(N)}$ be the standard approximation to Brownian motion. Compute $\lim_{N \rightarrow \infty} \mathbb{E}[e^{W_8^{(N)}}]$. Don't just set this up as an integral; compute the integral.

c. (5 pts.) Assume that f is four-times differentiable and that f'''' is continuous. Assume, for all $x \in \mathbb{R}$, that $|f''''(x)| \leq 7$. Let p be the third-order Maclaurin approximation to f . Give the smallest possible upper bound for $|(f(-3)) - (p(-3))|$.

d. (5 pts.) Assume that a train has jerk bounded by 5 mphph at all times. Assume that it leaves the station at zero velocity and zero acceleration. How many miles can it travel in 2 hours? That is, if $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 0$ and if $f''' \leq 5$, find the smallest possible upper bound for $f(2)$.

e. (5 pts.) Compute $\int_{1+2i}^{3+4i} e^{3x} dx$.

f. (5 pts.) Let $A := \int_{1+2i}^{3+2i} e^{x^2} dx$. Let $B := \int_{3+2i}^{3-2i} e^{x^2} dx$. Let $C := \int_{3-2i}^{1-2i} e^{x^2} dx$. Let $D := \int_{1-2i}^{1+2i} e^{x^2} dx$. Compute $A + B + C + D$.

g. (5 pts.) Compute $\int_{1+2i}^{3+4i} \bar{x} dx$, where \bar{x} denotes the complex conjugate of x .

h. (5 pts.) Compute $\int_{-\infty}^{\infty} e^{2x} e^{-x^2/2} dx$.

i. (5 pts.) Let Y be a PCRV such that $\Pr[Y = 4] = 0.25$, $\Pr[Y = 8] = 0.75$. Let $W := Y - (\mathbb{E}[Y])$.

(•) Compute the Fourier transform $\phi(t)$ of the distribution of W .

(•) Compute $\phi'(0)$.

j. (5 pts.) Let $\phi(t)$ be the Fourier transform described in the last problem (3.i. above). Compute $\lim_{N \rightarrow \infty} [\phi(t/\sqrt{N})]^N$.