FM 5002 Spring 2009, Final Exam
Handout date: Wednesday 6 May 2009

PRINT NAME:

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

SIGN NAME:

1. Definitions: Complete the following sentences.

a. (10 pts.) Let $f, g : \mathbb{R}^n \to \mathbb{R}$, let $k \in \mathbb{R}$ and let $C := \{x \in \mathbb{R}^n \mid g(x) = k\}$. A point $p \in C$ is said to be a **critical point** (for f on C) if there exists $\lambda \in \mathbb{R}$ such that ...

b. (10 pts.) A **vector field** on $\mathbb{R}^n$ is ...

c. (10 pts.) Let V be a vector field on $\mathbb{R}^n$ and let $c : (a, b) \to \mathbb{R}^n$ be smooth. We say that c is a **flowline** for V if ...

d. (10 pts.) A series $a_1 + a_2 + a_3 + \cdots$ is said to **converge** if ...

e. (10 pts.) Let $\Omega := [0, 1]$. Let X be a PCRV. Let $S := X(\Omega)$ be the set of values of X. Then the **partition** of X is ...

f. (10 pts.) Let $\Omega := [0,1]$. Let X be a PCRV, and let $\mathcal{P}$ be a partition of Ω by finite unions of intervals. Then $\mathrm{E}[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in A \in \mathcal{P}$ (and if A is not of zero size), then $(\mathrm{E}[X|\mathcal{P}])(\omega) = \cdots$

g. (10 pts.) Let $\Omega := [0,1]$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of Ω. We say that $\mathcal{Q}$ is **finer** than $\mathcal{P}$ if ...

h. (10 pts.) Let $\Omega := [0,1]$. Two events $A, B \subseteq \Omega$ are **independent** if ...

i. (10 pts.) Two PCRVs $X, Y \subseteq \Omega$ are **independent** if ...

j. (10 pts.) A PCRV X is said to be **standard** if ...

2. True or False. (No partial credit.)

a. (10 pts.) Let X be a PCRV. Then $\mathrm{SD}[-2X] = (-2)(\mathrm{SD}[X])$.

b. (10 pts.) Let X and Y be any two standard PCRVs. Then $(X+Y)/\sqrt{2}$ is again standard.

c. (10 pts.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and let $p \in \mathbb{R}^n$. Let $L := L_{f'(p)}$ and let $Q := Q_{f''(p)}$. Assume that $L = 0$ and that Q is positive definite. Then f attains a local minimum at p.

d. (10 pts.) Let X and Y be PCRVs. Assume that X and Y have the same distribution. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function. Then $\mathrm{E}[\phi(X)] = \mathrm{E}[\phi(Y)]$.

e. (10 pts.) Let V be a vector field on $\mathbb{R}^2$. Then there is a flowline $c : (-\infty, \infty) \to \mathbb{R}^2$ for V, footed at $(0,0)$, which is defined on the entire real line $\mathbb{R} = (-\infty, \infty)$.

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PLEASE DO NOT WRITE BELOW THE LINE

1.

2.

3a.

3b.

3c.

3d.

3ef.

3gh.

3ij.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (15 pts.) Stirling's formula asserts that $n! \sim (n/e)^n \sqrt{2\pi n}$. Find $a, b, c \in \mathbb{R}$ such that

$$\binom{3n}{n} \sim c a^n n^b.$$

b. (15 pts.) Let $C_1, C_2, \ldots$ be our standard iid sequence of standard binary variables (modeling infinite coin-flipping). For all integers $n \geq 1$, let $Z_n := (C_1 + \cdots + C_n)/\sqrt{n}$. Compute $\lim_{n\to\infty} \mathrm{E}[(e^{5Z_n} - e^{10})_+]$.

c. (15 pts.) Find the maximimum value of $96x + y$ subject to the constraint $3x^6 + y^6 = 193$.

d. (15 pts.) Compute $\displaystyle\int_{(1,1,1)}^{(2,2,2)} 3x\,dx - e^y\,dy + xy\,dz.$

e. (15 pts.) Let

$$C := \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad M := \begin{bmatrix} 6 & 5 & 23 \\ 5 & 6 & 21 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} 11 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$C^{-1} = \begin{bmatrix} 1/2 & 1/2 & 2 \\ 1/2 & -1/2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad D = C^{-1}MC.$$

Using all this, find expressions x, y and z (all three should be expressions of t) such that $[(x, y, z)]_{t:=0} = (1, 1, 1)$ and such that:

$$\begin{array}{rcrcrcr} dx/dt & = & 6x & + & 5y & + & 23z \\ dy/dt & = & 5x & + & 6y & + & 21z \\ dz/dt & = & 0 & + & 0 & + & 0. \end{array}$$

f. (15 pts.) Using your solution to 3e on the last page, find expressions x and y (both should be expressions of t) such that $[(x, y)]_{t:=0} = (1, 1)$ and such that:

$$\begin{array}{rcrcrcr} dx/dt & = & 6x & + & 5y & + & 23 \\ dy/dt & = & 5x & + & 6y & + & 21. \end{array}$$

g. (15 pts.) Using your solution to 3f on the last page, find the reverse gradient flowline for the functional

$$f(x, y) = 3x^2 + 5xy + 3y^2 + 23x + 21y + 7,$$

footed at $(1, 1)$. (IMPORTANT: Give the REVERSE gradient flowline, NOT the gradient flowine.)

h. (15 pts.) Let ω be the one-form in the single variable x defined by

$$\omega = [(2x^3 + e^{\ln x})^{100} - 2e^{x^2}(\sec^2(x^5)) + \tan^3((x^9)(\ln x))]\, dx.$$

Compute $d\omega$.

i. (15 pts.) Find the gradient of f, where $f(x, y, z) = e^{xz}[\sin(xz)]$.

j. (15 pts.) Let F be the expression of x, y and z given by $F = e^{xz}[\sin(xz)]$. Compute dF, the exterior derivative of the 0-form F.