FM 5002 Spring 2009, Midterm #2
Handout date: Wednesday 8 April 2009

PRINT NAME:

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

SIGN NAME:

1. Definitions: Complete the following sentences.

a. (5 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smoooth function. The **gradient** of f is the vector field $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(\nabla f)(x, y) = (\dots, \dots)$.

b. (5 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smoooth function. The **Hessian** of f is the matrix-valued function $Hf : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ defined by $(Hf)(x, y) = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$.

c. (5 pts.) Let X be a PCRV, and let $A \subseteq [0, 1]$ be finite union of intervals. Assume that A has positive size. Then $\mathrm{E}[X|A] = \cdots$.

d. (5 pts.) Let X be a PCRV, and let $\mathcal{P}$ be a partition of Ω by finite unions of intervals. Then $\mathrm{E}[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in A \in \mathcal{P}$ (and if A is not of zero size), then $(\mathrm{E}[X|\mathcal{P}])(\omega) = \cdots$

2. True or False. (No partial credit.)

a. (5 pts.) If p is the second-order Maclaurin approximation to f, then p and f have the same two-jet at 0.

b. (5 pts.) If X and Y are any two independent PCRVs, then $\mathrm{E}[X|Y]$ is deterministic.

c. (5 pts.) Let X and Y be PCRVs. Assume that X and Y have the same distribution. Then $X = Y$ a.s.

d. (5 pts.) Let X and Y be PCRVs. Assume that $X = Y$ a.s. Then X and Y have the same distribution.

e. (5 pts.) For any matrix $A \in \mathbb{R}^{2\times 2}$, there exists a matrix $B \in \mathbb{R}^{2\times 2}$ such that $BB^t = A$.

THE BOTTOM OF THIS PAGE IS FOR TOTALING SCORES
PLEASE DO NOT WRITE BELOW THE LINE

1.

2.

3ab.

3cd.

3ef.

3g.

3hi.

3j.

3k.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Let X be a binary PCRV such that $\Pr[X = 6] = 0.7$ and $\Pr[X = -14] = 0.3$. Compute the Fourier transform of the distribution of X.

b. (5 pts.) Let $X_1, X_2, \ldots$ be independent identically distributed binary PCRVs such that, for all integers $j \in [1, 100]$, both $\Pr[X_j = 6] = 0.7$ and $\Pr[X_j = -14] = 0.3$. Compute the Fourier transform of the distribution of $(X_1 + \cdots + X_{100})/10$.

c. (5 pts.) Let f be a function satisfying three conditions: First, f is continuous on $[0, 1]$. Second, $f(0) = f'(0) = f''(0) = 0$. Third, for all $t \in (0, 1)$, we have $f'''(t) \leq 10$. Among all such f, what is the maximum possible value of $f(1)$.

d. (5 pts.) Let g be a smooth function such that, for all $t \in [0, 1]$, we have $g'''(t) \leq 10$. Let p be the second order Maclaurin approximation of g. Compute an upper bound for $(g(1)) - (p(1))$.

e. (5 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sin(xe^y)$. Compute the gradient and Hessian of f.

f. (5 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sin(xe^y)$. What is the second order Maclaurin approximation to f?

g. (5 pts.) Let $f(x) = x^3 + (\cos(2x))$. Compute $\lim_{n\to\infty}[f(x/\sqrt{n})]^n$.

h. (5 pts.) How many terms are there in the 10th order Maclaurin approximation of a real-valued function of 7 variables (counting *all* terms, including those with a zero coefficient)?

i. (5 pts.) Compute $\int_{-\infty}^{\infty} x^{99} e^{-x^2/2}\, dx$.

j. (5 pts.) Compute $\int_{-\infty}^{\infty} (e^{3x} - e^3)_+ e^{-x^2/2}\, dx$.

k. (5 pts.) Let $W_t^{(N)}$ be the standard approximation to Brownian motion. Compute $\lim_{N\to\infty} \mathrm{E}[(W_6^{(N)})^4]$. Don't just set this up as an integral; compute the integral.