FM 5011 Fall 2007, Midterm #2
Handout date: Thursday 8 November 2007

1. Definitions: Complete the following sentences.

a. (5 pts.) Two random variables X and Y are said to be **independent** if ...

b. (5 pts.) Let X be a square integrable random variable and let $m := \mathrm{E}[X]$. The **variance** of X is the expectation of ...

(Your answer should be a simple formula involving X and m.)

c. (5 pts.) Let μ be a measure on (the Borel sets of) $\mathbb{R}$. The **cumulative distribution function** of μ is the function $f : \mathbb{R} \to [0, 1]$ defined by $f(x) = \cdots$

d. (5 pts.) A measure space X is said to be **σ-finite** if ...

2. True or False. (No partial credit.)

a. (5 pts.) If X and Y are any two integrable random variables, then $\mathrm{E}[X + Y] = (\mathrm{E}[X]) + (\mathrm{E}[Y])$.

b. (5 pts.) If X and Y are any two square integrable random variables, then $\mathrm{E}[XY] = (\mathrm{E}[X])(\mathrm{E}[Y])$.

c. (5 pts.) If X is any random variable, if F is the cumulative distribution function of X, if $s, t \in \mathbb{R}$ and if $s < t$, then $F(s) < F(t)$.

d. (5 pts.) For any two events A and B (*i.e.*, for any two subsets $A, B \subseteq [0, 1]$), we have the following equality of conditional probabilities: $\Pr[A|B] = \Pr[B|A]$.

e. (5 pts.) For any two L^2 random variables, we have $(\mathrm{Covar}[X, Y])^2 \leq \mathrm{Var}[X] \cdot \mathrm{Var}[Y]$.

f. (5 pts.) If two random variables are independent, then they are uncorrelated.

g. (5 pts.) If two random variables are uncorrelated, then they are independent.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} e^{7x} e^{-x^2/2}\, dx$.

b. (5 pts.) Suppose A and B are two events (*i.e.*, $A, B \subseteq [0,1]$ are both measurable). Suppose $\Pr[A] = 0.2$, $\Pr[B] = 0.3$ and $\Pr[A \text{ or } B] = 0.4$. Compute $\Pr[A|B]$.

c. (5 pts.) Let X and Y be two standard random variables. (Thus, $\mathrm{E}[X] = 0$, $\mathrm{E}[Y] = 0$, $\mathrm{SD}[X] = 1$ and $\mathrm{SD}[Y] = 1$.) Assume that $\mathrm{Corr}[X,Y] = 0.3$. Compute $\mathrm{SD}[X+Y]$.

d. (5 pts.) Let X, Y and Z be independent random variables. Assume $\Pr[X = 1] = 0.7$ and $\Pr[X = 2] = 0.3$. Assume that Y and Z have the same distribution as X. Compute $\mathrm{E}[X+Y+Z]$ and $\mathrm{Var}[X+Y+Z]$.

e. (5 pts.) Let Y be a normal variable with mean 1 and variance 2. Compute the following two expected values: $\mathrm{E}[Y^2]$ and $\mathrm{E}[e^{4Y+7}]$.

f. (5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} [2e^x - 5]_+\, e^{-x^2/2}\, dx$.

4. Miscellaneous.

a. (5 pts.) Let $X_1, X_2, \ldots$ be iid random variables, all L^2, all with mean 0 and all with variance 8. For all integers $n > 0$, let $Y_n := (X_1 + \cdots + X_n)/\sqrt{n}$. Compute $\displaystyle\lim_{n\to\infty} \mathrm{E}[e^{Y_n}]$.

b. (5 pts.) Let W be an L^2 random variable with mean 2 and variance 3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mathrm{E}[(W - x)^2]$. Minimize f. That is, find x_0 such that, for all $x \in \mathbb{R}$, $f(x_0) \le f(x)$.

c. (5 pts.) Let X be the grade of a standard normal random variable. Compute $\mathrm{SD}[X]$.