FM 5011 Fall 2007, Final Exam
Handout date: Thursday 6 December 2007

1. Definitions: Complete the following sentences.

a. (5 pts.) A **σ-algebra** on a set X is a collection $\mathcal{S}$ of subsets of X such that ...

b. (5 pts.) Two random variables X and Y are said to be **independent** if ...

c. (5 pts.) The **Δ** of an option is the following difference quotient: the ... difference over the ... difference.

d. (5 pts.) Let X be a square integrable random variable, and let $m := \mathrm{E}[X]$. The **standard deviation** of X is ...

(Your answer should be a simple formula involving X and m.)

e. (5 pts.) Let X be a random variable, and let $\mathcal{S}$ be a σ-algebra on $[0, 1]$. We say that X is **measurable** with respect to $\mathcal{S}$ (or **$\mathcal{S}$-measurable**) if ...

f. (5 pts.) A **process** is a measurable map from ... to ...

g. (5 pts.) Let X be a random variable, and let $\mathcal{F}$ be a σ-algebra. The **conditional expectation** of X with respect to $\mathcal{F}$, denoted $\mathrm{E}[X|\mathcal{F}]$, is a random variable Y such that the following conditions hold: ...

h. (5 pts.) Let X be a process. We say that X is a **martingale** if the following conditions hold: ...

i. (5 pts.) Let X be a piecewise continuous process with step Δt. Then $\Delta X_t = \cdots$

2. True or False. (No partial credit.)

a. (5 pts.) If X and Y are any two integrable random variables, then
$$\mathrm{E}[X+Y] = (\mathrm{E}[X]) + (\mathrm{E}[Y]).$$

b. (5 pts.) If X is a random variable, and if F is the cumulative distribution function of X, then F is continuous.

c. (5 pts.) For any two square integrable random variables X and Y, we have
$$2(\mathrm{Covar}[X,Y]) = (\mathrm{Var}[X+Y]) - (\mathrm{Var}[X]) - (\mathrm{Var}[Y]).$$

d. (5 pts.) Let X satisfy the SDE
$$dX_t = (0.25)X_t\,dW_t + (0.07)X_t\,dt.$$
Then there exists a change of measure Q such that $\tilde{X} := X^Q$ satisfies
$$d\tilde{X}_t = (0.25)\tilde{X}_t\,dW_t + (0.05)\tilde{X}_t\,dt.$$

e. (5 pts.) Let Z be a standard normal variable. Then $e^{Z^2/2}$ is L^1.

f. (5 pts.) For any two events A and B (*i.e.*, for any two subsets $A, B \subseteq [0,1]$), we have the following equality of conditional probabilities: $(\Pr[A])(\Pr[A|B]) = (\Pr[B])(\Pr[B|A])$.

g. (5 pts.) If two random variables X and Y are uncorrelated, then, for any two Borel functions $f, g : \mathbb{R} \to \mathbb{R}$, the random variables $f(X)$ and $g(Y)$ are uncorrelated.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations.)

a. (5 pts.) Compute $\int_7^\infty e^{4x+3}e^{-x^2/2}\,dx$.

b. (5 pts.) Compute $\int_7^\infty e^{4x+3}e^{-x^2}\,dx$.

c. (5 pts.) Let X satisfy the SDE

$$dX_t = (0.25)X_t\,dW_t + (0.07)X_t\,dt$$

and let $Y := \ln X$. Using Itô's Lemma, find the SDE satisfied by Y.

d. (5 pts.) Let X, Y and Z be random variables. Assume

$$\mathrm{Corr}[X,Y] = \mathrm{Corr}[X,Z] = \mathrm{Corr}[Y,Z] = 0.2.$$

Assume $\Pr[X = 2] = 0.6$ and $\Pr[X = 3] = 0.4$. Assume that Y and Z have the same distribution as X. Compute $\mathrm{Var}[X + Y + Z]$.

e. (5 pts.) Let Y be a normal variable with mean 1 and variance 2. Compute the following two expected values: $\mathrm{E}[Y^2]$ and $\mathrm{E}[e^{4Y+7}]$.

f. (5 pts.) Suppose X has mean 0.5 and variance $(0.2)^2$. Suppose Y has mean 0.2 and variance $(0.3)^2$. Suppose $\mathrm{Corr}[X,Y] = 0.5$ Find constants a and b such that the mean of $aX+bY$ is 2.5 the variance of $aX+bY$ is minimized. (Note: a and/or b might be negative.)

g. (5 pts.) Let X and Y be jointly normal. (This means, for some independent standard normal random variables, Z_1 and Z_2, that both X and Y are linear combinations of Z_1 and Z_2.) Assume that both X and Y have mean zero and variance 1. Also, assume that $\mathrm{Corr}[X,Y] = 1/\sqrt{2}$. Let $W := \mathrm{E}[X|Y]$. Find $\mathrm{Var}[W]$.

(Hint: You may assume $Y = Z_1$ and $X = (Z_1/\sqrt{2}) + (Z_2/\sqrt{2})$.)

4. Let X satisfy the ("Ornstein-Uhlenbeck") SDE $dX_t = dW_t - (0.5)X_t\,dt$, with initial condition $X_0 = 1$. Define Y by $Y_t = e^{(0.5)t}\,X_t$. Let $S := Y^2$. The following sequence of problems leads the calculation of the mean and variance of X_t, at time $t = 9$.

Hint: You may use the fact that, if P satisfies an SDE of the form $dP_t = \sigma_t\,dW_t + f(t)\,dt$, where σ is adapted to W_t, and where $f(t)$ is a deterministic function of t, then $(d/dt)(\mathrm{E}[P_t]) = f(t)$.

a. (5 pts.) Using the product rule, find the SDE satisfied by Y.

b. (5 pts.) Using Itô's Lemma, find the SDE satisfied by S.

c. (5 pts.) Find $\mathrm{E}[Y_9]$.

d. (5 pts.) Find $\mathrm{E}[S_9]$.

e. (5 pts.) Find $\mathrm{Var}[Y_9]$.

f. (5 pts.) Find $\mathrm{E}[X_9]$.

g. (5 pts.) Find $\mathrm{Var}[X_9]$.