FM 5011 Fall 2008, Final exam
Handout date: Thursday 18 December 2008

PRINT NAME:

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

SIGN NAME:

1. Definitions: Complete the following sentences.

1a. (5 pts.) Let M be a set and let $\mathcal{B}$ be a σ-algebra on M. Then a **measure** on $(M, \mathcal{B})$ is a function $\mu : \mathcal{B} \to [0, \infty]$ such that ...

1b. (5 pts.) Let X and Y be two L^2 random variables. Then $\mathrm{Corr}[X, Y] := \cdots$

1c. (5 pts.) Let $(M, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{Z} := \{B \in \mathcal{B} \,|\, \mu(B) = 0\}$. Let $\mathcal{Z}_1$ be the collection of all subsets of sets in $\mathcal{Z}$. Then the **completion** of $\mathcal{B}$ with respect to μ is ...

1d. (5 pts.) Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Let X be a random variable. We say that $X_n \to X$ **in probability** as $n \to \infty$ if ...

1e. (5 pts.) Let $\mathbb{T}$ be a set. A $\mathbb{T}$-**process** is ...

1f. (5 pts.) A **process** is ...

1g. (5 pts.) Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $\mathbb{T} \subseteq \mathbb{R}$. A function

$$\mathcal{F}_\bullet : \mathbb{T} \to \{\sigma\text{-subalgebras of } \mathcal{B}\}$$

is called a **filtration** if ...

1h. (5 pts.) Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. A random variable $X : \Omega \to \mathbb{R}$ is said to be L^2 if ...

2. True or False. (No partial credit.)

2a. (5 pts.) Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Let X and Y be random variables. Assume both that $X_n \to X$ in distribution and that $X_n \to Y$ in distribution, as $n \to \infty$. Then $X = Y$ a.s.

2b. (5 pts.) Let $W_\bullet$ be a Brownian motion. Then, for all $t \geq 0$, $\mathrm{Var}[W_t] = t$.

2c. (5 pts.) For all pairs X and Y of independent random variables,

$$\mathrm{SD}[X + Y] = (\mathrm{SD}[X]) + (\mathrm{SD}[Y]).$$

2d. (5 pts.) For any two measures μ and ν on a Borel space $(X, \mathcal{B})$, there exists a measurable function $f : X \to [0, \infty]$ such that $\nu = f\mu$.

2e. (5 pts.) Let $W_\bullet$ be a Brownian motion on a probability space $(\Omega, \mathcal{B}, \mu)$. Then, for any change of measure ν, the process $W_\bullet$ is again a ν-Brownian motion.

2f. (5 pts.) Any L^2 random variable is L^1.

2g. (5 pts.) Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Let $f_1, f_2, \ldots : \Omega \to [0,1]$ be a sequence of measurable functions and let $f : \Omega \to [0,1]$ be a measurable function. Assume, for all $\omega \in \Omega$, that $f_n(\omega) \to f(\omega)$, as $n \to \infty$. Then

$$\int_\Omega f_n \, d\mu \to \int_\Omega f \, d\mu, \qquad \text{as } n \to \infty.$$

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PLEASE DO NOT WRITE BELOW THE LINE

1.

2.

3a.

3b.

3cd.

3ef.

3gh.

3ij.

3kl.

4ab.

4cd.

4efg.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations.) In any of the exercises in this part of the exam, $W_\bullet$ denotes a Brownian motion.

3a. (5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} (e^{5x+4} - e^{14})_+ e^{-x^2/2}\, dx.$

3b. (5 pts.) Let X and Y be independent PCRVs such that $\Pr[X = 3] = \Pr[Y = 3] = 0.4$ and such that $\Pr[X = 7] = \Pr[Y = 7] = 0.6$. Write $\mathrm{dstr}[X + Y]$ as a convex combination of delta masses. (A "convex combination" is a linear combination in which both the coefficients are all ≥ 0 and the sum of the coefficients is 1.)

3c. (5 pts.) Let $Y_1, \ldots, Y_{100}$ be independent identically distributed random variables, each of which is L^2. Let $X := Y_1 + \cdots + Y_{100}$. Assume that X has mean 0.07 and variance 0.2. Compute the mean and variance of Y_1.

3d. (5 pts.) For every integer $n \geq 1$, let X_n be a sum of n independent identically distributed random variables, each of which is L^2. Assume, for all integers $n \geq 1$, that X_n has mean 0.07 and variance 0.2. Compute $\lim_{n\to\infty} \mathrm{E}[e^{X_n}]$.

3e. (5 pts.) Compute $\mathrm{E}[W_1 W_2]$.

3f. (5 pts.) Let $X := \int_0^T t^2 \, dW_t$ and let $Y := \int_0^T t^3 \, dW_t$. Compute $\mathrm{Covar}[X, Y]$.
(*Hint*: Compute $\mathrm{Var}[X]$, $\mathrm{Var}[Y]$ and $\mathrm{Var}[X + Y]$. Then compute $\mathrm{Covar}[X, Y]$.)

3g. (5 pts.) Let $X_\bullet$ solve $dX_t = (0.22)\,dW_t + (0.04)\,dt$ and let $S_\bullet := e^{X_\bullet}$. Using Itô's Lemma, compute dS_t/S_t.

3h. (5 pts.) Suppose $Y_\bullet = \sin^2(X_\bullet)$. Following Itô's Lemma, write a formula for dY_t in terms of X_t and dX_t. (Your answer may involve "$(dX_t)^2$".)

3i. (5 pts.) Solve the SDE $dX_t/X_t = 3\,dW_t + 4\,dt$ with initial value $X_0 = 1$.

3j. (5 pts.) Let $X_\bullet$ solve the SDE above $dX_t/X_t = 3\,dW_t + 4\,dt$, with initial value $X_0 = 1$. Write $\mathrm{E}[(X_3 - 7)_+]$ as a definite integral, from $-\infty$ to ∞, of some expression of x (against dx). You need not compute the integral; just set it up.

3k. (5 pts.) Let $X_\bullet$ solve $dX_t/X_t = 2e^t\, dW_t + t^2\, dt$. Find a process $\gamma_\bullet$ such that, if $W_\bullet$ solves $d\tilde{W}_t = dW_t + \gamma_t\, dt$, then $dX_t/X_t = 2e^t\, d\tilde{W}_t + 0.03\, dt$.

3l. (5 pts.) Compute $\mathrm{E}\left[\int_0^3 (e^{3t-7} - 7)_+ e^{-t^2/2}\, dW_t\right]$. (WARNING: Note that the integral is not against dt, but, rather, against dW_t.)

4. Let $W_\bullet$ be a Brownian motion. Let $X_\bullet$ satisfy the ("Ornstein-Uhlenbeck, mean-reverting") SDE $dX_t = dW_t - (0.3)X_t\,dt$, with initial condition $X_0 = 1$. Define $Y_\bullet$ by $Y_t = [e^{(0.3)t}][X_t]$. Let $S_\bullet := (Y_\bullet)^2$. The following sequence of problems leads the calculation of the mean and variance of X_t, at time $t = 9$.

4a. (5 pts.) Using the product rule, find the SDE satisfied by $Y_\bullet$. Your answer may involve dW_t and dt.

4b. (5 pts.) Using Itô's Lemma, find a formula for dS_t in terms of Y_t, dW_t and dt. Your answer should *not* involve dY_t.

4c. (5 pts.) Find $\mathrm{E}[Y_9]$. (*Hint*: Integrate the SDE of 4a, and then take expectation at $t = 9$. Remember that $X_0 \neq 0$; in fact, $X_0 = 1$.)

4d. (5 pts.) Find $\mathrm{E}[S_9]$. (*Hint*: Integrate the SDE of 4b, and then take expectation at $t = 9$. Remember that $S_0 \neq 0$.)

4e. (5 pts.) Find Var[Y_9]. (*Hint*: Note that Problem 4d gives E[$(Y_9)^2$] and that Problem 4c gives E[Y_9].)

4f. (5 pts.) Find E[X_9]. (*Hint*: Note that Problem 4c is closely related. Review how X_t and Y_t are related.)

4g. (5 pts.) Find Var[X_9]. (*Hint*: Note that Problem 4e is closely related. Review how X_t and Y_t are related.)