FM 5011 Fall 2008, Midterm #2
Handout date: Thursday 13 November 2008

PRINT NAME:

1. Definitions: Complete the following sentences.

a. (5 pts.) A subset $S \subseteq \mathbb{R}^n$ is said to be **measurable** if ...

b. (5 pts.) Let $(M, \mathcal{B}, \mu)$ be a measure space. Let $(K, \mathcal{C})$ be a Borel space. A function $f : M \to K$ is said to be **Borel** if ...

c. (5 pts.) Let $(M, \mathcal{B}, \mu)$ be a measure space. Let $(K, \mathcal{C})$ be a Borel space. A function $f : M \to K$ is said to be **measurable** if ...

d. (5 pts.) Let $(M, \mathcal{B}, \mu)$ be a measure space. Let $(K, \mathcal{C})$ be a Borel space. Let $f : M \to K$ be measurable. Then $f_*(\mu)$ is defined by the rule: for all $C \in \mathcal{C}$, $(f_*(\mu))(C) = \cdots$.

e. (5 pts.) Let μ be a measure on $\mathbb{R}$. The **cumulative distribution function** of μ is the function $F : \mathbb{R} \to [0, \infty]$ defined by $F(x) = \cdots$.

2. True or False. (No partial credit.)

a. (5 pts.) If $(M, \mathcal{B}, \mu)$ is a measure space, then μ has a unique extension to a measure on the completion of $\mathcal{B}$ w.r.t. μ.

b. (5 pts.) The cumulative distribution function of a measure on $\mathbb{R}$ is always left continuous.

c. (5 pts.) If a series converges, then all of its rearrangements converge.

d. (5 pts.) The Borel spaces $[0, 1]$ and $\mathbb{R}^3$ (with their usual σ-algebras) are Borel isomorphic.

e. (5 pts.) The measure spaces $[0, 1]$ and $\mathbb{R}^3$ (with their usual σ-algebras and measures) are measure space isomorphic.

f. (5 pts.) Let K be a set, let $\mathcal{A} \subseteq 2^K$ and let $\mathcal{B}$ be the σ-algebra on K generated by $\mathcal{A}$. Let $\mu, \nu : \mathcal{B} \to [0, \infty]$ be two measures on $(K, \mathcal{B})$. Assume, for all $A \in \mathcal{A}$, that $\mu(A) = \nu(A)$. Then, for all $B \in \mathcal{B}$, we have $\mu(B) = \nu(B)$.

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1.

2.

3ab.

3c.

3de.

3fg.

3hi.

3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct, and can be left in any form easily calculated on a calculator. No partial credit, except in unusual situations.)

a. (5 pts.) Let Y be a binary random variable such that

$$\Pr[Y = 3] = 0.2 \qquad \text{and} \qquad \Pr[Y = 10] = 0.8.$$

Let $F : \mathbb{R} \to [0, 1]$ be defined by $F(x) = \Pr[Y \leq x]$. Let $G := F \circ Y$. Compute $\Pr[G < 0.2]$.

b. (5 pts.) Let $\Omega := (0, 1)$, with (restricted) Lebesgue measure. Let $Z := \Phi^{-1} : (0, 1) \to \mathbb{R}$. Let $Y := Z^2$. Define $F : \mathbb{R} \to [0, 1]$ by $F(x) = \Pr[Y \leq x]$. Let $G := F \circ Y : (0, 1) \to [0, 1]$. Compute $\Pr[0.3 \leq G \leq 0.7]$.

c. (5 pts.) Let $I := [0, 1]$, as a measure space, with (restricted) Lebesgue measure, denoted λ_1. Let $\Omega := I \times I \times I \times \cdots$, with the (infinite) product measure, denoted $\mu := \lambda_1 \times \lambda_1 \times \cdots$. Let S be the set of all those points $(x_1, x_2, \ldots) \in \Omega$, satisfying: for some $j \geq 7$, $x_j < 0.5$.

Problem c1: Find a sequence of basic rectangles $R_1, R_2, \ldots$ such that $\mu(R_j) \to 0$ and such that, for all j, $\Omega \backslash R_j \subseteq S$.

Problem c2: Compute $\mu(S)$.

d. (5 pts.) Let $f(x) = x^6$. Let $Y_1, Y_2, \ldots$ be an i.i.d. sequence of standard PCRVs. For all integers $n \geq 1$, let

$$Z_n := \frac{Y_1 + \cdots + Y_n}{\sqrt{n}}.$$

Compute $\lim_{n\to\infty} \mathrm{E}[f(Z_n)]$.

e. (5 pts.) Let $\mathcal{B}$ be the σ-algebra on $\mathbb{R}$ generated by $(-\infty, 7]$, $(6, 9]$, $(8, 11]$ and $(10, \infty)$.

Problem e1: Find all the atoms in $\mathcal{B}$.

Problem e2: How many sets are there in $\mathcal{B}$?

f. (5 pts.) Define a PCRV X by:

$$X = 4 \text{ on } [0, 0.3] \qquad \text{and} \qquad X = 6 \text{ on } (0.3, 1].$$

Let λ_1 be the restriction of Lebesgue measure to $[0, 1]$. Compute $X_*(\lambda_1)$ as a linear combination of delta masses. (Recall that the delta mass at a, denoted δ_a, is defined by

$$\delta_a(S) = 1 \text{ if } a \in S \qquad \text{and} \qquad \delta_a(S) = 0 \text{ if } a \notin S. \qquad)$$

g. (5 pts.) Let $Z := \Phi^{-1} : (0, 1) \to \mathbb{R}$. Let $f(x) = (e^x - e^7)_+$. Let $Y := f \circ Z : (0, 1) \to \mathbb{R}$. Compute $\displaystyle\int_0^1 Y(\omega)\, d\omega$. (*Hint*: Make the change of variables $\omega = \Phi(x)$.)

h. (5 pts.) Give an example of two standard binary random variables that do not have the same distribution.

i. (5 pts.) Give an example of two PCRVs X and Y and two functions $f, g : \mathbb{R} \to \mathbb{R}$ with the following properties: The level sets of X are $[0, 0.5]$ and $(0.5, 1]$. The level sets of Y are $[0, 0.25]$, $(0.25, 0.5]$ and $(0.5, 1]$. Also, $\mathrm{E}[X] = \mathrm{E}[f \circ X] = 0$, $\mathrm{E}[XY] = 0$ and $\mathrm{E}[(f \circ X)(g \circ Y)] \neq 0$.