Financial Mathematics

Matrix operations

0023-1. Compute

$$\begin{bmatrix} -1 & 2 & 0 & -9 \\ 5 & 2 & 4 & -1 \\ -3 & 6 & -4 & 9 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \\ -2 & 0 & 2 \end{bmatrix}.$$

0023-2. Compute

$$\begin{bmatrix} 0 & 3 & 6 & 9 \\ 3 & -2 & 9 & 12 \\ -1 & 8 & -3 & -4 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -5 & 8 \\ 2 & 1 & -3 & 0 \\ 7 & 0 & 1 & 6 \end{bmatrix}.$$

0023-3. Find the transpose of

$$\begin{bmatrix} -2 & -3 & -4 & -5 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

0023-4. Compute

$$\begin{bmatrix} -1 & 3 & 0 & -7 \\ 2 & -2 & 7 & 3 \\ 4 & 3 & -4 & 6 \end{bmatrix} \oplus \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \\ 1 & 4 & 7 \end{bmatrix}.$$

0023-5.

$$A := \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -3 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Compute $A \otimes B$ and $B \otimes A$.

0023-6. Find the left conjugate of

$$\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{bmatrix} \text{ by } \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

0023-7. Let $A := \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ and $B := \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$ and $C := \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$.

a. Compute AB and BA.
b. Compute ACB.
c. Compute e^C.
d. Compute e^{ACB}.

Parts c and d belong in Topic 0024.

e. Compute $[2] \oplus [3]$.
f. Compute $A \oplus C$
g. Compute $A \otimes C$

0023-8. Let $A := \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}$ and $B := \begin{bmatrix} -5 & 1 \\ 0 & -7 \end{bmatrix}$.

a. Compute $A + B$.

b. Compute $B + A$.

c. Compute AB.

d. Compute BA.

e. Compute $A \oplus B$.

f. Compute $B \oplus A$.

g. Compute $A \otimes B$.

h. Compute $B \otimes A$.

0023-9.

Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$

$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

Let $A_{11} = \begin{bmatrix} 6 & 7 \\ 8 & -7 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -7 & 3 \\ 2 & 6 \end{bmatrix},$

$A_{21} = \begin{bmatrix} 6 & 5 \\ 3 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 0 \\ -5 & 8 \end{bmatrix}.$

Compute $(E_{11} \otimes A_{11}) + (E_{12} \otimes A_{12}) +$
$(E_{21} \otimes A_{21}) + (E_{22} \otimes A_{22})$.

0023-10. Let E_{11}, E_{12}, E_{21} and E_{22} be as in 0023-9.

Let $B_{11} := \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$, $B_{12} := \begin{bmatrix} 9 & 7 \\ 8 & 2 \end{bmatrix}$,

$B_{21} := \begin{bmatrix} 6 & -2 \\ 5 & 9 \end{bmatrix}$, $B_{22} := \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$.

Find C_{11}, C_{12}, C_{21} and C_{22} such that

$$(E_{11} \otimes C_{11}) + (E_{12} \otimes C_{12}) + (E_{21} \otimes C_{21}) + (E_{22} \otimes C_{22}) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

0023-11. Let E_{11}, E_{12}, E_{21} and E_{22} be as in 0023-9.

Define $\mathcal{M} : (\mathbb{R}^{2\times 2})^{2\times 2} \to \mathbb{R}^{2\times 2}$ by the rule:
$\forall X_{11}, X_{12}, X_{21}, X_{22} \in \mathbb{R}^{2\times 2}$,

$$\mathcal{M}((E_{11} \otimes X_{11}) + (E_{12} \otimes X_{12}) + (E_{21} \otimes X_{21}) + (E_{22} \otimes X_{22})) = E_{11}X_{11} + E_{12}X_{12} + E_{21}X_{21} + E_{22}X_{22}$$

Let $A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\quad B := \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix}$

Compute $\mathcal{M}(A \otimes B)$ and AB.

Hint: Note that

$$A = E_{11} + 2E_{12} + 3E_{21} + 4E_{22}.$$

0023-12.

$$M := \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}, \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\forall a, b, c, d \in \mathbb{R}$, ← commutative

$$\text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc, \qquad \text{t-cof} \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\forall a, b, c, d \in \mathbb{R}$,
$\forall p, q, r, s \in \mathbb{R}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

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a. Compute

$$(\text{t-cof}\, M)M, \;\; M(\text{t-cof}\, M), \;\; (\det M)I.$$

b. Show, $\forall K \in \mathbb{R}^{2\times 2}$, that

$$(\text{t-cof}\, K)K = K(\text{t-cof}\, K) = (\det K)I.$$

Hint: Write $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and compute.

0023-13. $M := \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}$, $X := \begin{bmatrix} 2 & 7 \\ -3 & -8 \end{bmatrix}$, $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\forall A, B, C, D \in \{$linear combinations of $I, X, X^2, \ldots\}$, ← commutative

$$\boxed{\text{DET}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} := AD - BC, \qquad \boxed{\text{T-COF}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}.$$

$\forall A, B, C, D \in \mathbb{R}X$,
$\forall P, Q, R, S \in \mathbb{R}X$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix}.$$

$$\mathcal{M} := M \otimes X = \begin{bmatrix} 7X & 5X \\ 6X & 4X \end{bmatrix}.$$

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a. Compute
$(\text{T-COF}\,\mathcal{M})\mathcal{M}$, $\mathcal{M}(\text{T-COF}\,\mathcal{M})$, $I \otimes [\text{DET}\,\mathcal{M}]$.

b. Show, $\forall \mathcal{K} \in (\mathbb{R}X)^{2\times 2}$, that
$(\text{T-COF}\,\mathcal{K})\mathcal{K} = \mathcal{K}(\text{T-COF}\,\mathcal{K}) = I \otimes [\text{DET}\,\mathcal{K}]$.

Hint: Captialize all the letters from 0023-12b.