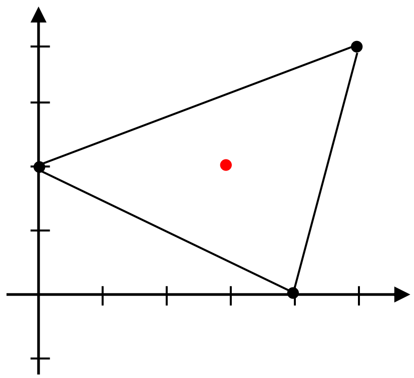
Financial Mathematics

Variations on Stokes' Theorem

$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad$ $\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

0042-1. a. Compute $\displaystyle\int_{2i}^{5+4i} \cos(2z)\,dz$.



b. Compute $\displaystyle\int_{5+4i}^{4} \cos(2z)\,dz$.

c. Compute $\displaystyle\int_{2i}^{4} \cos(2z)\,dz$.

d. Show that the complex limit

$$\lim_{h\to 0} \frac{\cos(2(3+2i+h)) - \cos(2(3+2i))}{h} \text{ exists,}$$

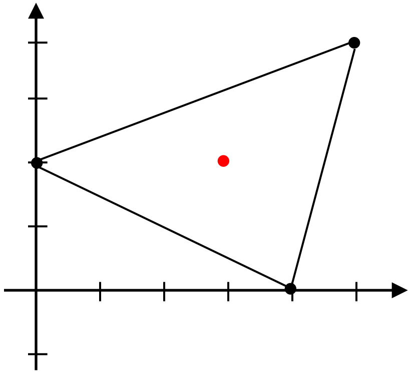
and is equal to $-2\sin(2(3+2i))$.

0042-2. a. Compute $\displaystyle\int_{2i}^{5+4i} |z|^4\,dz$.

b. Compute $\displaystyle\int_{5+4i}^{4} |z|^4\,dz$.

c. Compute $\displaystyle\int_{2i}^{4} |z|^4\,dz$.

d. Show that the complex limit



$$\lim_{h\to 0} \frac{|3+2i+h|^4 - |3+2i|^4}{h}$$

does not exist.

0042-3. Let $P := p(x,y) = -y$.
Let $Q := q(x,y) = x$.
Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $V(x,y) = (P,Q)$. (The "tornado".)
Let $\alpha(t) = (5, 3+t)$, $\beta(t) = (5-3t, 4)$, $\gamma(t) = (2, 4-t)$, $\delta(t) = (2+3t, 3)$, for $0 \le t \le 1$.
Compute
$$\left[\int_0^1 [V(\alpha(t))]\cdot[\alpha'(t)]\,dt\right] + \left[\int_0^1 [V(\beta(t))]\cdot[\beta'(t)]\,dt\right] + \left[\int_0^1 [V(\gamma(t))]\cdot[\gamma'(t)]\,dt\right] + \left[\int_0^1 [V(\delta(t))]\cdot[\delta'(t)]\,dt\right].$$
(To what extent does the current help us as we swim around α, β, γ, then δ.)

unassigned:

Let $P := p(x,y) = -y$.

Let $Q := q(x,y) = x$.

Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $V(x,y) = (P,Q)$. (The "tornado".)

Let R be a rectangle in the plane.

Let α, β, γ and δ parametrize the four sides of R, in the counterclockwise direction.

Show that $\left[\int_0^1 [V(\alpha(t))] \cdot [\alpha'(t)]\,dt\right] +$

$$\left[\int_0^1 [V(\beta(t))] \cdot [\beta'(t)]\,dt\right] + \left[\int_0^1 [V(\gamma(t))] \cdot [\gamma'(t)]\,dt\right] +$$

$\left[\int_0^1 [V(\delta(t))] \cdot [\delta'(t)]\,dt\right]$ equals twice the area of R.

(The extent to which the current helps us is the area enclosed in our swim.)

0042-4. Let $P := p(x,y) = -y$.
Let $Q := q(x,y) = x$.

Let $\alpha(t) = (5, 3+t)$, $\beta(t) = (5-3t, 4)$,
$\gamma(t) = (2, 4-t)$, $\delta(t) = (2+3t, 3)$,
for $0 \le t \le 1$.

Let $a := \alpha(0) = \delta(1)$. Let $b := \beta(0) = \alpha(1)$.
Let $c := \gamma(0) = \beta(1)$. Let $d := \delta(0) = \gamma(1)$.

Let K be the 1-chain $\{(a,b),(b,c),(c,d),(d,a)\}$.
Let $\omega := P\,dx + Q\,dy$.

Compute $\displaystyle\int_K \omega$.

0042-5. Let $P := p(x,y) = -y$.
Let $Q := q(x,y) = x$.

Let R be the rectangle $[2,5] \times [3,4]$.

Let $\omega := P\,dx + Q\,dy$.

a. Compute ∂R.

b. Compute $\displaystyle\int_{\partial R} \omega$.

c. Compute $\displaystyle\int_{R} d\omega$.

Let $i := \sqrt{-1}$.

0042-6. Let $P := p(x,y) = x^2 - y^2$.
Let $Q := q(x,y) = 2xy$.
Let $\alpha(t) = (5, 3+t)$, $\beta(t) = (5-3t, 4)$,
$\gamma(t) = (2, 4-t)$, $\delta(t) = (2+3t, 3)$,
for $0 \leq t \leq 1$.

Let $a := \alpha(0) = \delta(1)$. Let $b := \beta(0) = \alpha(1)$.
Let $c := \gamma(0) = \beta(1)$. Let $d := \delta(0) = \gamma(1)$.

Let K be the 1-chain $\{(a,b),(b,c),(c,d),(d,a)\}$.

Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x+iy) = P + iQ$
(Note that $f(z) = z^2$.)
Let $\omega := f(z)\,dz$.

Compute $\displaystyle\int_K \omega$.

Let $i := \sqrt{-1}$.

0042-7. Let $P := p(x,y) = x^2 - y^2$.
Let $Q := q(x,y) = 2xy$.

Let $\alpha(t) = 5 + (3+t)i$, $\beta(t) = (5-3t) + 4i$,
$\gamma(t) = 2 + (4-t)i$, $\delta(t) = (2+3t) + 3i$,
for $0 \le t \le 1$.

Let $a := \alpha(0) = \delta(1)$. Let $b := \beta(0) = \alpha(1)$.
Let $c := \gamma(0) = \beta(1)$. Let $d := \delta(0) = \gamma(1)$.
Let K be the 1-chain $\{(a,b),(b,c),(c,d),(d,a)\}$.

Compute $\displaystyle\int_K (P + iQ)(dx + i\,dy)$.

i.e., compute $\displaystyle\left[\int_K P\,dx\right] - \left[\int_K Q\,dy\right] + i\left[\int_K Q\,dx\right] + i\left[\int_K P\,dy\right]$

0042-8. Compute $\int_{(7,8,9)}^{(9,8,7)} x\,dx + ye^x\,dy - x^2e^z\,dz$,

i.e., compute $\int_L x\,dx + ye^x\,dy - x^2e^z\,dz$,

where L is the directed line segment from $(7,8,9)$ to $(9,8,7)$.

0042-9. Let $R := (2,4) \times (6,9)$.

Compute $\int_R [e^{2x-3y}]\,dy \wedge dx$.

0042-10.

Let $\omega = f(u,x,y,z,t)$ be any 0-form in u,x,y,z,t.

Show that $d(d\omega) = 0$.

0042-11. Let A, B, C, P, Q, R, E, F, G, H be expressions in u,x,y,z,t.

Let $$\omega := A\,dx\wedge dy + B\,dx\wedge dz + C\,dy\wedge dz + P\,dx\wedge dt + Q\,dy\wedge dt + R\,dz\wedge dt + E\,du\wedge dx + F\,du\wedge dy + G\,du\wedge dz + H\,du\wedge dt.$$

so ω is a 2-form in u,x,y,z,t.

Show that $d(d\omega) = 0$.

0042-12. $i := \sqrt{-1}$

Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^z + \sin\left(|z|^2\right)$.

Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x + iy) = [u(x, y)] + [v(x, y)]i.$$

Let $U := u(x, y)$ and $V := v(x, y)$.

a. Find U and V.

b. Determine whether the

$\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$ Cauchy-Riemann equations

hold.

$\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ $\qquad i := \sqrt{-1}$

0042-13. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \sin(z^2)$.

Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x + iy) = [u(x, y)] + [v(x, y)]i.$$

Let $U := u(x, y)$ and $V := v(x, y)$.

a. Find U and V.

b. Determine whether the

$\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$ Cauchy-Riemann equations

hold.

$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $i := \sqrt{-1}$

0042-14. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2 \cos(z)$.

Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x + iy) = [u(x,y)] + [v(x,y)]i.$$

Let $U := u(x,y)$ and $V := v(x,y)$.

a. Find U and V.

b. Determine whether the

$\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$ Cauchy-Riemann equations

hold.

$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $i := \sqrt{-1}$

0042-15. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \cos(\overline{z})$.

Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x + iy) = [u(x, y)] + [v(x, y)]i.$$

Let $U := u(x, y)$ and $V := v(x, y)$.

a. Find U and V.

b. Determine whether the

$\partial_x U = \partial_y V$ & $\partial_x V = -\partial_y U$ Cauchy-Riemann equations

hold.

0042-16. Write

$$[(xz+y^3)dx+3yz\,dy+(z^4+xyz)\,dz]$$
$$\wedge\ [(8-2xyz^3)\,dx+(4y-3)\,dy+(y^7-x)\,dz]$$

in the form $[f(x,y,z)]\,dx\wedge dy$
$$+[g(x,y,z)]\,dx\wedge dz$$
$$+[h(x,y,z)]\,dy\wedge dz$$

0042-17. Write

$$[xy^2z^3\,dx\wedge x^3y^2z\,dy+dx\wedge dz+(y^2+2xyz)\,dy\wedge dz]$$
$$\wedge\ [(z-2y)\,dx+4x^2y^2z^2\,dy+(5z-2xy)\,dz]$$

in the form $[f(x,y,z)]\,dx\wedge dy\wedge dz$

0042-18. Compute $d(x^2 e^{yz})$,
the exterior derivative of

$$x^2 e^{yz}$$

with respect to x, y, z.

0042-19.
Compute $d(y^5\,dx + 2e^{xyz}\,dy - 4e^{-xz}\,dz)$,
the exterior derivative of

$$y^5\,dx + 2e^{xyz}\,dy - 4e^{-xz}\,dz$$

with respect to x, y, z.