Financial Mathematics

Numbers and sets of numbers

Factorial notation

Def'n: $0! = 1$

$\forall$integers $n \geq 1$,

$n! = n(n-1)(n-2)\cdots(3)(2)(1)$

"n factorial"

e.g.: $1! = 1$, $2! = (2)(1) = 2$,

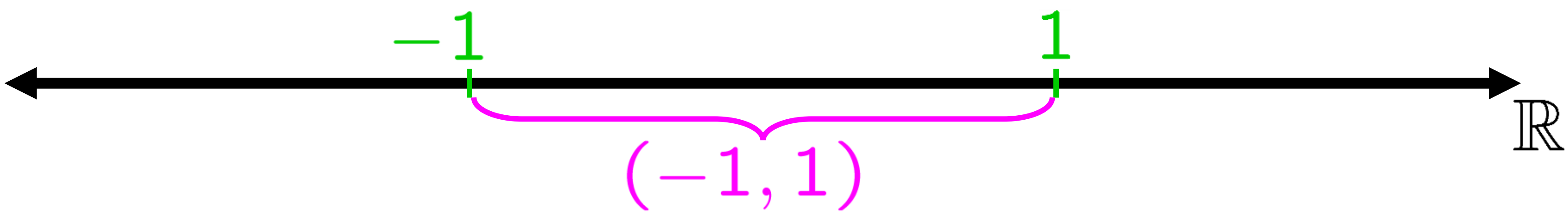
$3! = (3)(2)(1) = 6$, $4! = (4)(3)(2)(1) = 24$,

$5! = (5)(4)(3)(2)(1) = 120$

Intervals

"the open interval from -1 to 1"

$$(-1,1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$$



$\mathbb{R} :=$ {real numbers} "="

Warning:

$(-1,1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$

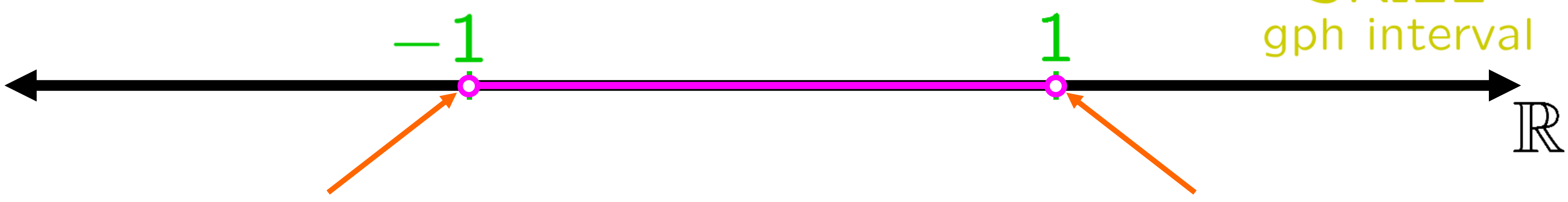
$(-1,1)$ is an ordered pair.

Not equal!!

Intervals

$$(-1,1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$$

SKILL
gph interval



$\mathbb{R} := \{\text{real numbers}\}$ "="

Warning:

Not equal!!

$(-1,1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$

$(-1,1)$ is an ordered pair.

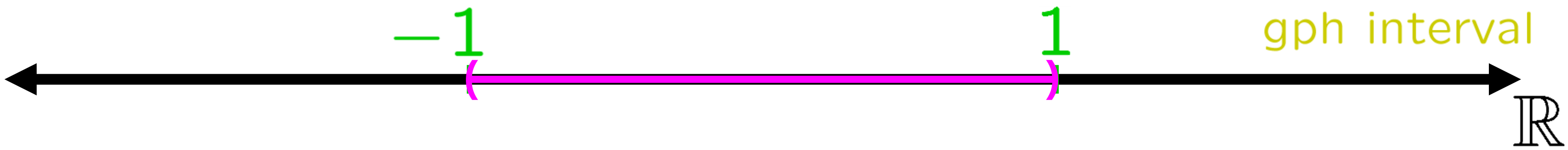
Intervals

(finite) numbers

$(-1, 1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$

open, bounded

SKILL
gph interval



$\mathbb{R} :=$ {real numbers} "=" ⟷

Warning:

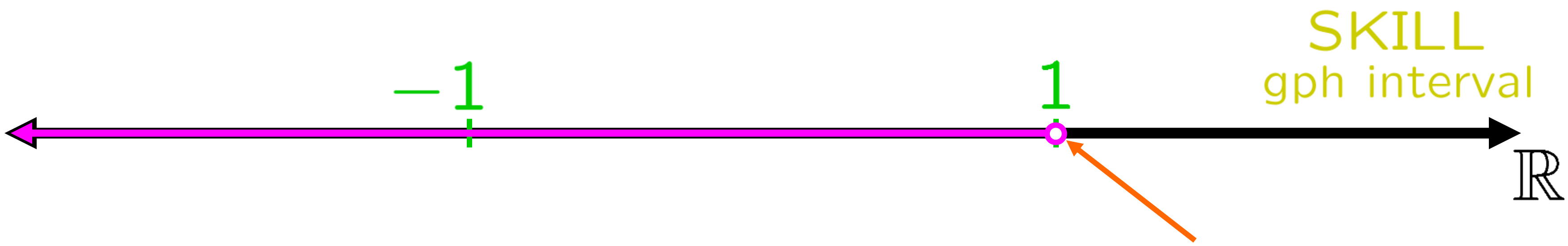
Not equal!!

$(-1, 1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$

$(-1, 1)$ is an ordered pair.

Intervals

$$(-\infty, 1) = \{x \in \mathbb{R} \,|\, x < 1\}$$



$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ "=" 

Intervals

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open, unbounded

SKILL
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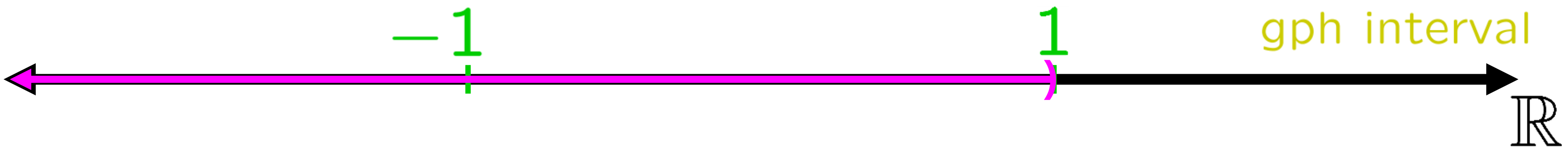
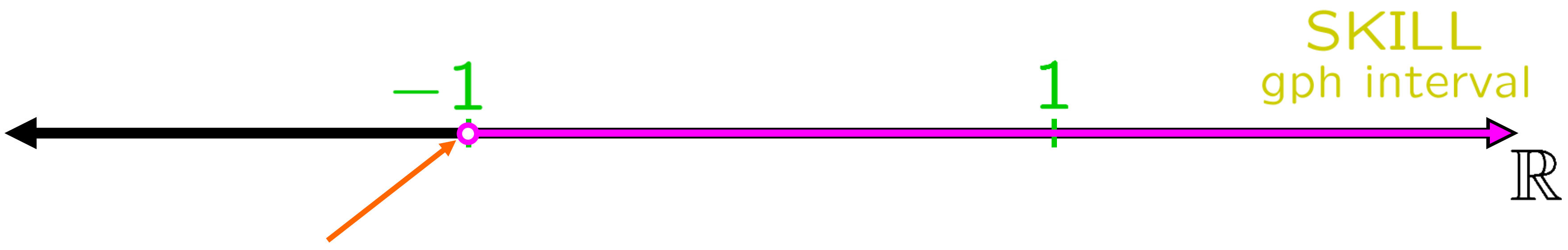


$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ "=" 

Intervals

$$(-1, \infty) = \{x \in \mathbb{R} \mid -1 < x\}$$



$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ "$=$"

Intervals

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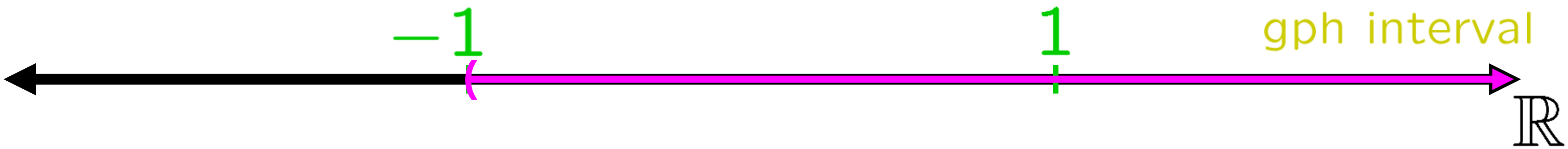
open, unbounded

SKILL
gph interval



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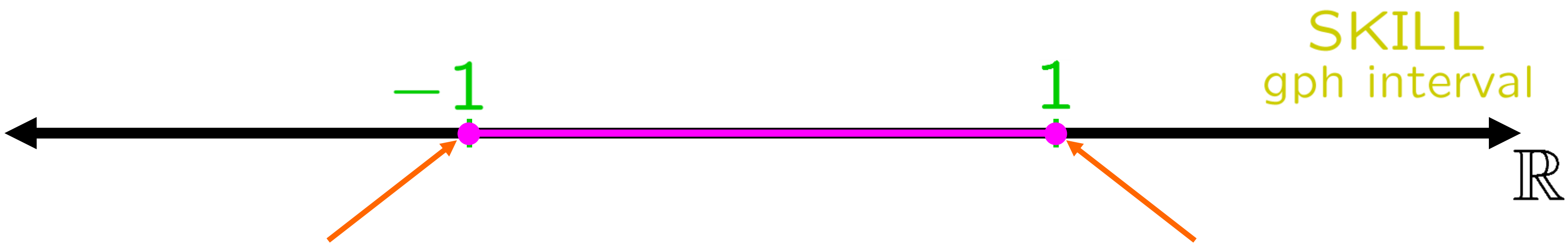
Intervals

$$[-1,1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$$



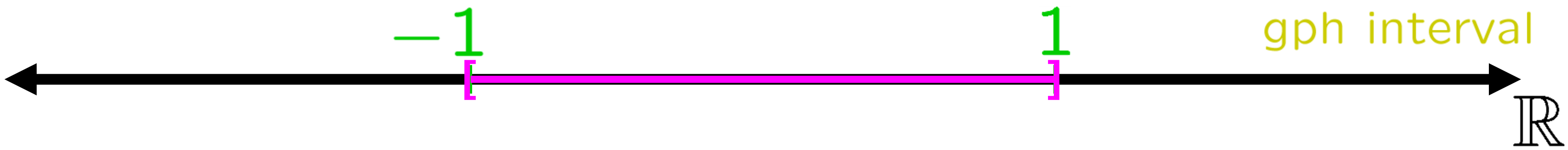
$\mathbb{R} := \{\text{real numbers}\}$ "$=$"

Intervals

$$[-1,1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$$

closed, bounded $=:$ **compact**

SKILL
gph interval



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SKILL
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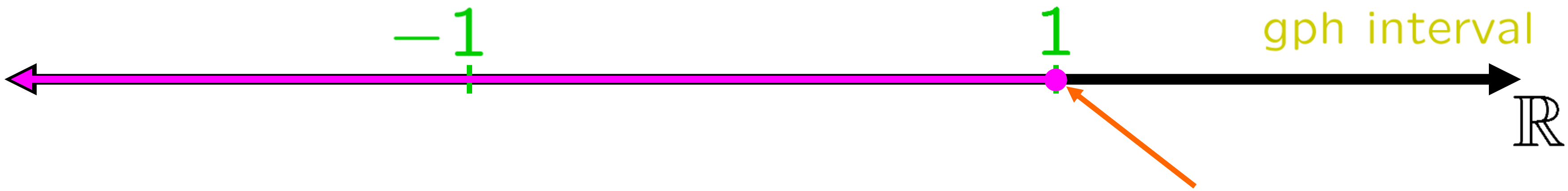


$\mathbb{R}$:= {real numbers} "=" 

Intervals

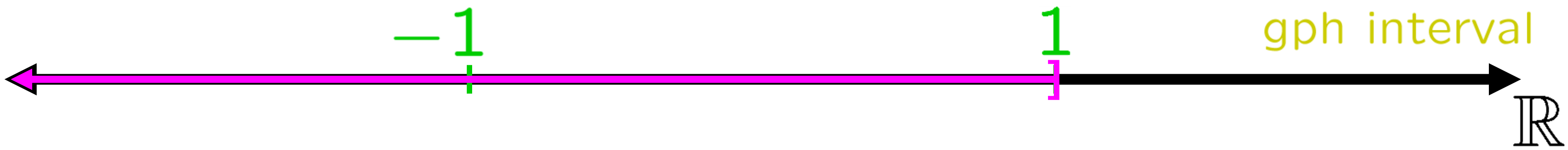
Intervals of the form $[a, b]$,
with $-\infty < a < b < \infty$,
are said to be **compact**, *i.e.*, closed and bounded.

$$(-\infty, 1] = \{x \in \mathbb{R} \,|\, x \leq 1\}$$

CLOSED, unbounded, noncompact

as closed as possible

SKILL
gph interval



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Intervals

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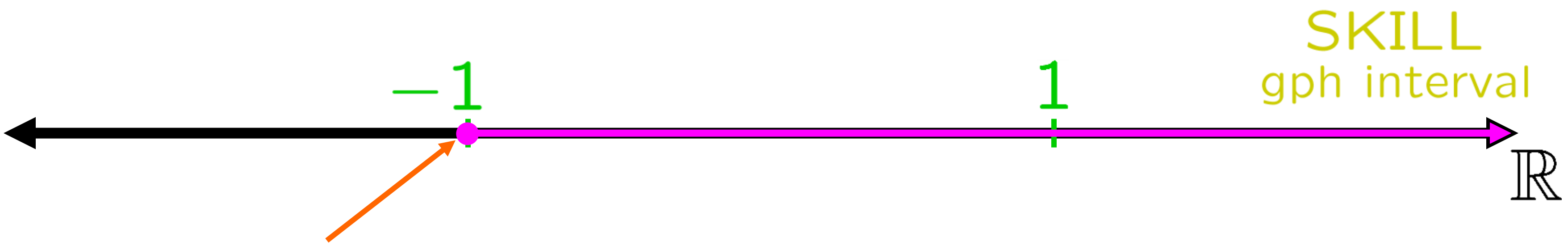

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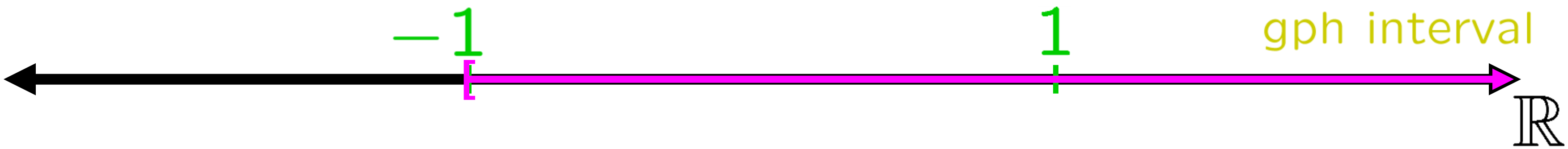

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SKILL
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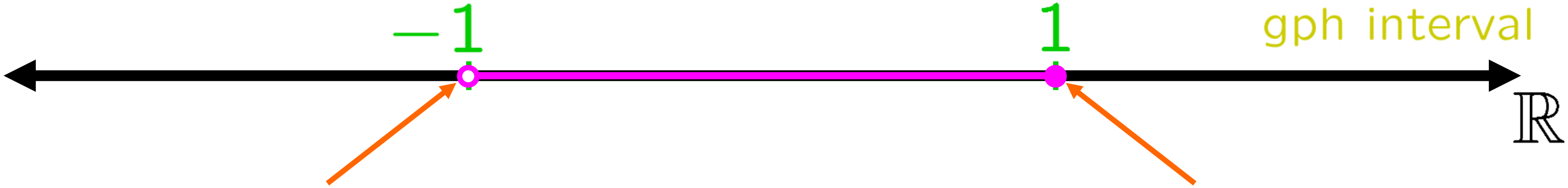


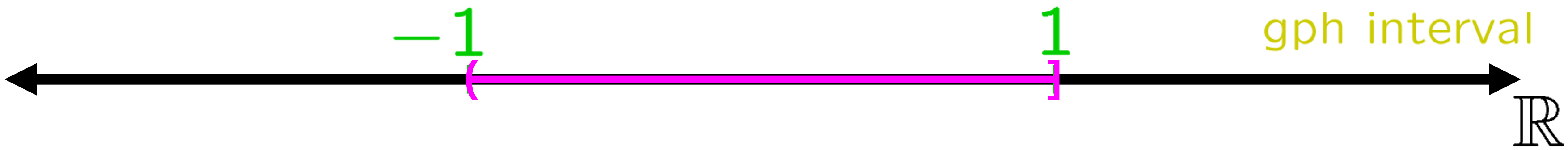
$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ "$=$"

Intervals

$$(-1, 1] = \{x \in \mathbb{R} \mid -1 < x \leq 1\}$$

half-open, bounded

SKILL
gph interval

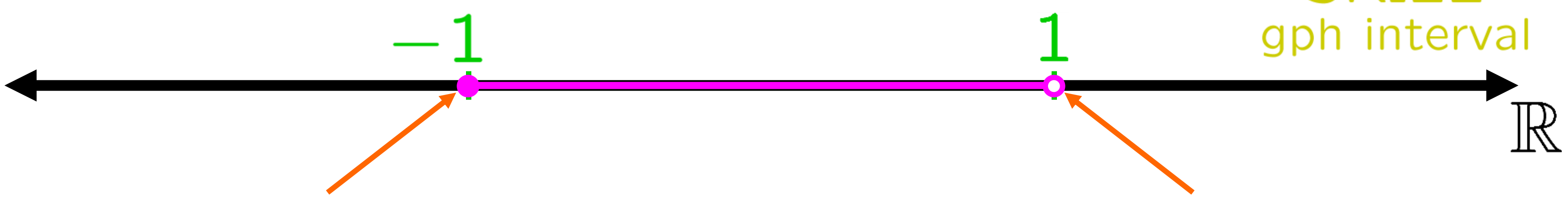


$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ "$=$"

Intervals

$$[-1,1) = \{x \in \mathbb{R} \mid -1 \leq x < 1\}$$

SKILL
gph interval



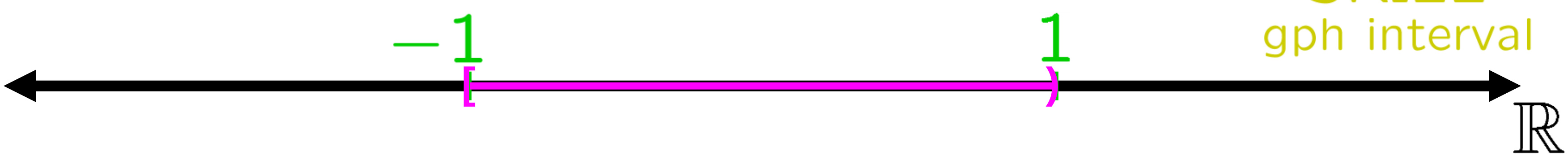
$\mathbb{R}$:= {real numbers} "="

Intervals

$$[-1, 1) = \{x \in \mathbb{R} \mid -1 \leq x < 1\}$$

half-open, bounded

SKILL
gph interval



$\boxed{\mathbb{R}} := \{\text{real numbers}\}$ “=”

Intervals

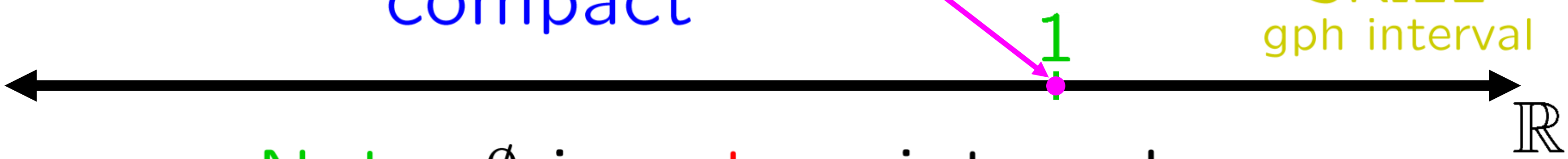
An interval is **degenerate**
if it has only one element.

$$[1,1] = \{1\}$$

CLOSED, bounded, degenerate

$\underbrace{\text{CLOSED, bounded}}_{\text{compact}}$

SKILL
gph interval



Note: $\emptyset$ is not an interval.

$\mathbb{R} := \{\text{real numbers}\}$ "="

Intervals

An interval is **degenerate**
if it has only one element.

SKILL
identify interval:
open, closed, half-open,
bounded, unbounded
degenerate, non-degenerate

Note: Half-open intervals are
neither open, nor closed.

Note: $\mathbb{R} = (-\infty, \infty)$ is
both open and closed.

Note: $\emptyset$ is not an interval.

---

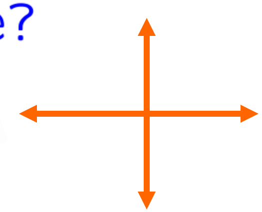
picture?

$\mathbb{Z} := \{\text{integers}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{R} := \{\text{real numbers}\}$ "$=$" 

$= \{\text{rationals}\} \cup \{\text{irrationals}\}$

$\mathbb{Q} := \{\text{rational numbers}\}$ picture?

$\mathbb{C} := \{\text{complex numbers}\}$ "$=$" 

Picture $\mathbb{Z}$

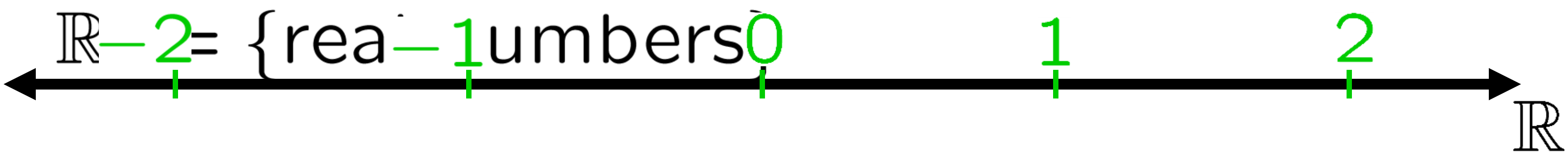
$\mathbb{R} := \{\text{real numbers}\}$

$\mathbb{Z} := \{\text{integers}\}$

picture?

$\mathbb{Z} := \{\text{integers}\}$

$\mathbb{R} := \{\text{real numbers}\}$



Picture $\mathbb{Z}$

$\mathbb{R} := \{\text{real numbers}\}$

has non-$\emptyset$ intersection with

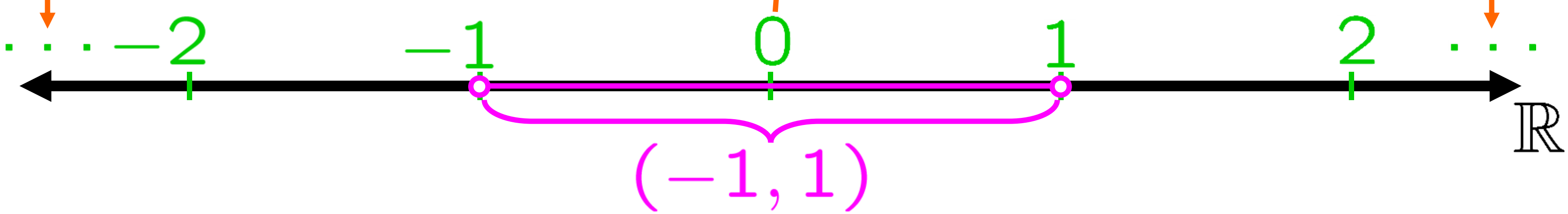
$\mathbb{Z}$ "meets" $(-1,1)$.

$\mathbb{Z} \cap (-1,1) \neq \emptyset$

$\cup$

0

$\mathbb{Z} := \{\text{integers}\}$



Picture $\mathbb{Z}$

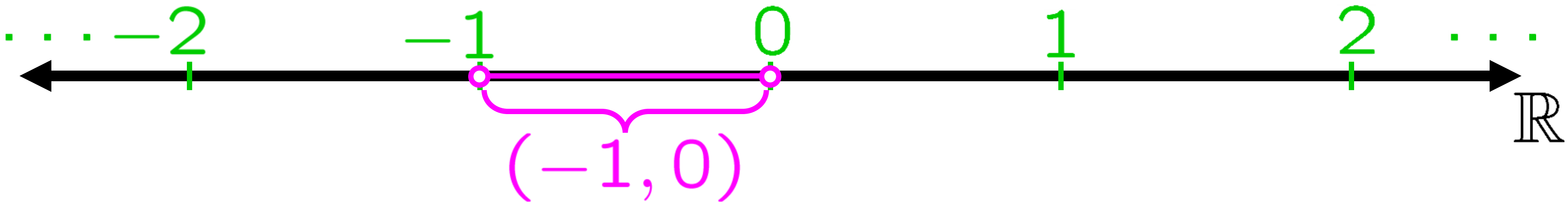
$\mathbb{R} :=$ {real numbers}

$\mathbb{Z} :=$ {integers}

$\mathbb{Z}$ "meets" $(-1,1)$.

$$\mathbb{Z} \cap (-1,1) \neq \emptyset$$
$$\mathbb{Z} \cap (-1,0) = \emptyset$$

$\mathbb{Z}$ is disjoint from $(-1,0)$.



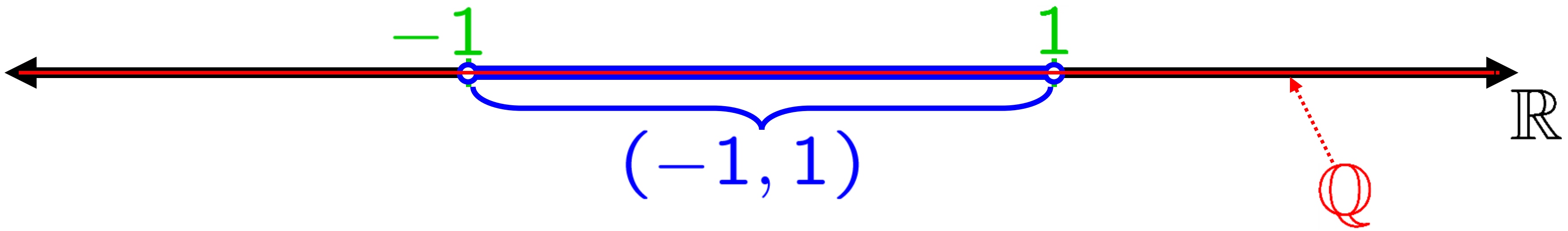
Picture $\mathbb{Q}$

$\mathbb{R} :=$ {real numbers}

$$\mathbb{Q} \cap (-1, 1) \neq \emptyset$$

$\mathbb{Q} :=$ {rational numbers}

$\mathbb{Z} :=$ {integers}



Picture $\mathbb{Q}$

has non-$\emptyset$ intersection with

meets every open interval

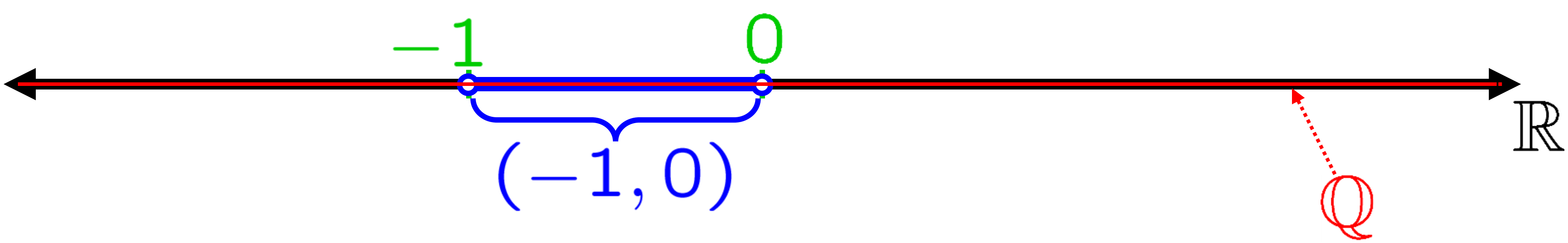
$\mathbb{R} := \{\text{real numbers}\}$

$\mathbb{Q} \cap (-1, 1) \neq \emptyset$

$\mathbb{Q} \cap (-1, 0) \neq \emptyset$

$\mathbb{Q} := \{\text{rational numbers}\}$ is dense in $\mathbb{R}$.

$\mathbb{Z} := \{\text{integers}\}$



Extended Real Numbers

Definition:
An **extended real number** is
a real number or $\underbrace{+\infty}_{\infty}$ or $-\infty$.

Definition: $\overline{\mathbb{R}} := \{\text{extended real numbers}\}.$

$$= \mathbb{R} \cup \{-\infty, \infty\}$$

Note:

For all $s \in \mathbb{R}$, $-\infty < s < \infty$.

For all $s \in \overline{\mathbb{R}}$, $-\infty \leq s \leq \infty$.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

$$-\infty \le a < b \le \infty \Rightarrow (a,b) := \{x \in \overline{\mathbb{R}} \mid a < x < b\}$$
$$-\infty \le a \le b \le \infty \Rightarrow [a,b] := \{x \in \overline{\mathbb{R}} \mid a \le x \le b\}$$
$$-\infty \le a < b \le \infty \Rightarrow (a,b] := \{x \in \overline{\mathbb{R}} \mid a < x \le b\}$$
$$-\infty \le a < b \le \infty \Rightarrow [a,b) := \{x \in \overline{\mathbb{R}} \mid a \le x < b\}$$

These sets are **extended intervals**.

An **interval** is an extended interval I s.t. $I \subseteq \mathbb{R}$.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

e.g.: $(0,\infty)$ is an interval. ($\subseteq \mathbb{R}$; unbounded)

$(0,\infty]$ is an extended interval, but is not an interval. ($\not\subseteq \mathbb{R}$)

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

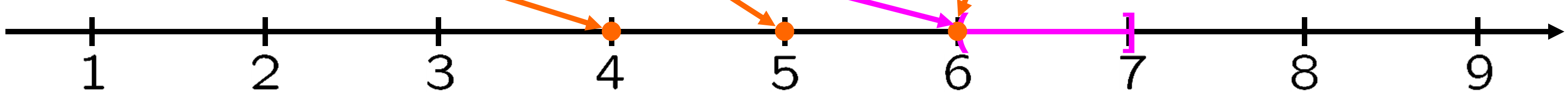
Def'n: Let $a \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say a is a **lower bound for** S,

and write $a \le S$,

if: $\forall s \in S$, we have $a \le s$.

e.g.: $4 \le (6,7]$, $5 \le (6,7]$, $6 \le (6,7]$



Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

greatest lower bound

We say b is the **inf** (or **glb**) **of** S,

infimum

and write $b = \inf S$,

if: $[b \le S]$ and $[\forall a \le S$, we have $a \le b]$.

e.g.: $6 = \inf\,(6,7]$

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

Def'n: Let $a \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

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e.g.: $6 = \inf (6,7]$

Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say b is the **inf** (or **glb**) **of** S,

inf = **infimum**; **glb** = **greatest lower bound**

and write $b = \inf S$,

if: $[b \leq S]$ and $[\forall a \leq S$, we have $a \leq b]$.

If, in addition, $b \in S$,

then we say that b is the **min of** S,

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

Def'n: Let $a \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say a is a **lower bound for** S,

Next: **upper bnd&max**

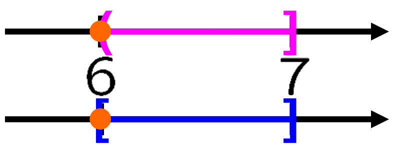
and write $a \leq S$,

if: $\forall s \in S$, we have $a \leq s$.

---

e.g.: $4 \leq (6, 7]$, $5 \leq (6, 7]$, $6 \leq (6, 7]$

$(6, 7]$ has no min.



e.g.: $6 = \inf\, (6, 7]$

e.g.: $6 = \min\, [6, 7]$

Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

greatest lower bound

We say b is the **inf** (**or glb**) **of** S,

infimum

and write $b = \inf S$,

if: $[b \leq S]$ and $[\forall a \leq S$, we have $a \leq b]$.

If, in addition, $b \in S$,

minimum

then we say that b is the **min of** S,

and write $b = \min S$.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

Def'n: Let $a \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

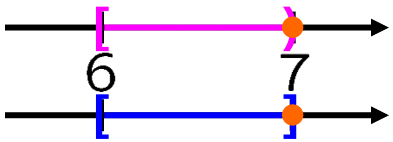
We say a is a **upper bound for** S,

and write $a \geq S$,

if: $\forall s \in S$, we have $a \geq s$.

e.g.: $9 \geq [6,7)$, $8 \geq [6,7)$, $7 \geq [6,7)$

$[6,7)$ has no max.



e.g.: $7 = \sup [6,7)$

e.g.: $7 = \max [6,7]$

Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say b is the **sup** (supremum) (or **lub** (least upper bound)) **of** S,

and write $b = \sup S$,

if: $[b \geq S]$ and $[\forall a \geq S$, we have $a \geq b]$.

If, in addition, $b \in S$,

then we say that b is the **max** (maximum) **of** S,

and write $b = \max S$.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

Def'n: Let $a \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say a is a **upper bound for** S,

and write $a \geq S$,

if: $\forall s \in S$, we have $a \geq s$.

---

e.g.: $\sup(2, \infty) = \infty$ $\quad$ $\inf(-\infty, \infty) = -\infty$

$\max(2, \infty)$ DNE $\quad$ $\min(-\infty, \infty)$ DNE

does not exist

$\max(2, \infty] = \infty$ $\quad$ $\min[-\infty, \infty) = -\infty$

Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

We say b is the **sup** (or **lub**) **of** S,

least upper bound

supremum

and write $b = \sup S$,

if: $[b \geq S]$ and $[\forall a \geq S$, we have $a \geq b]$.

If, in addition, $b \in S$,

then we say that b is the **max of** S,

maximum

and write $b = \max S$.

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

Theorem:

Every subset of $\overline{\mathbb{R}}$ has an inf and a sup (in $\overline{\mathbb{R}}$).

---

e.g.: $\sup(2,\infty) = \infty$ $\qquad$ $\inf(-\infty,\infty) = -\infty$

$\max(2,\infty)$ DNE (does not exist) $\qquad$ $\min(-\infty,\infty)$ DNE

$\max(2,\infty] = \infty$ $\qquad$ $\min[-\infty,\infty) = -\infty$

Def'n: Let $b \in \overline{\mathbb{R}}$, $S \subseteq \overline{\mathbb{R}}$.

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If, in addition, $b \in S$,

then we say that b is the **max** (maximum) of S, and write $b = \max S$.

$$-\infty \le a < b \le \infty \Rightarrow \boxed{(a,b)} := \{x \in \overline{\mathbb{R}} \,|\, a < x < b\}$$
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These sets are **extended intervals**.

An **interval** is an extended interval I s.t. $I \subseteq \mathbb{R}$.

Note: $\{1/2\} = [1/2, 1/2]$ is an interval.

(Sometimes called a **degenerate** interval.)

~~$(1/2, 1/2)$, $[1/2, 1/2)$ and $(1/2, 1/2]$~~

are not allowed.

non-e.g.:

$[0, 1/3] \coprod [2/3, 1]$ is not an interval,

although it is a finite union of intervals.

fUofI

$-\infty \leq a < b \leq \infty \Rightarrow (a,b) := \{x \in \overline{\mathbb{R}} \,|\, a < x < b\}$
$-\infty \leq a \leq b \leq \infty \Rightarrow [a,b] := \{x \in \overline{\mathbb{R}} \,|\, a \leq x \leq b\}$
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These sets are **extended intervals**.

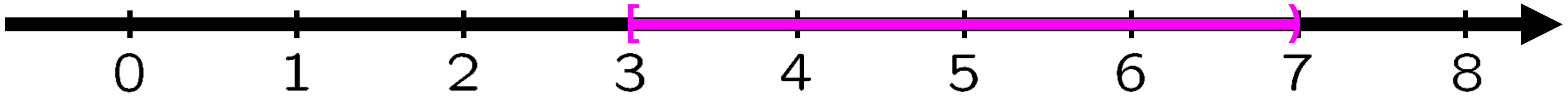
An **interval** is an extended interval I s.t. $I \subseteq \mathbb{R}$.

Definition: For any interval I,

the **length** of I is $|I| := (\sup I) - (\inf I)$.

e.g.: $|[3,7)| = 7 - 3 = 4$

$[3,\infty)$?



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Definition: For any interval I,

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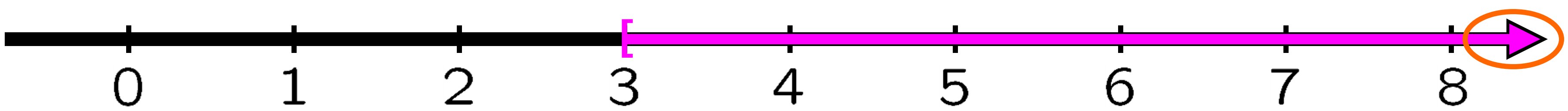
e.g.: $|[3,\infty)| = \infty - 3 = \infty$

For intervals, bounded $\Leftrightarrow$ finite length



Definition:

If $I_1, \ldots, I_n$ is a finite collection of pairwise disjoint intervals,

then $|I_1 \coprod \cdots \coprod I_n| := |I_1| + \cdots + |I_n|$

the **measure** of $I_1 \coprod \cdots \coprod I_n$

$\forall$ integers $j, k \in [1, n]$, if $j \neq k$, then $I_j \cap I_k = \emptyset$.

---

e.g.:

$$|[0, 1/3] \coprod [3/4, 7/8)| = |[0, 1/3]| + |[3/4, 7/8)|$$
$$= (1/3) + (1/8) = 11/24$$

Warning: ~~$\coprod$~~

$$|[0, 2/3] \cup [1/3, 1]| \neq |[0, 2/3]| + |[1/3, 1]|$$

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SKILL
measure of a fUofI

SKILL
express a fUofI as a
finite DISJOINT union of intervals

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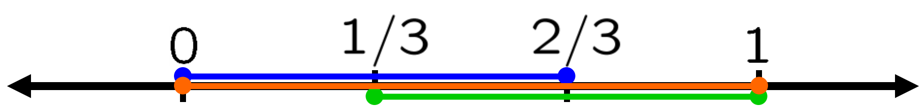

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Let $S \subseteq \mathbb{R}$. A partition of $\mathcal{P}$ of S is said to be a **partition by intervals** if:

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$$P \text{ is an interval.}$$

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both $\displaystyle\bigcup_{P \in \mathcal{P}} P = S$

and $\forall P, Q \in \mathcal{P}, \quad (P \neq Q \quad \Rightarrow \quad P \cap Q = \emptyset)$.

---

e.g.: $\{ \quad \{1\} \quad , \quad \{2,3,5\} \quad , \quad \{4,6\} \quad \}$ is a partition of $\{1,2,3,4,5,6\}$.

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non-e.g.:

$\{\ [0, 1/3] \cup [2/3, 1]\ ,\ (1/3, 2/3)\ \}$ is a partition of $[0, 1]$, but not by intervals.

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finite unions of intervals

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