Financial Mathematics

Absolute value and distance

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

absolute value of x

Definition: $\underset{5}{\boxed{|x|}} := \begin{cases} \overset{5}{x}, & \text{if } \overset{5}{x} \geq 0 \\ -x, & \text{if } \underset{5}{x} \leq 0 \end{cases}$

e.g.: $|5| = 5$ ■

SKILL
comp abs val

absolute value of x

Definition: $\underset{-5}{\boxed{|x|}} := \begin{cases} x, & \text{if } \overset{-5}{x} \ge 0 \\ -x, & \text{if } \underset{-5}{x} \le 0 \end{cases}$

$-(-5)$

e.g.: $|-5| = -(-5) = 5$ ■

SKILL
comp abs val

absolute value of x

Definition: $\underset{0}{\boxed{|x|}} := \begin{cases} \overset{0}{x}, & \text{if } \overset{0}{x} \geq 0 \\ -x, & \text{if } \underset{0}{x} \leq 0 \end{cases}$

$-\ 0$

e.g.: $|0| = 0$ ■

SKILL
comp abs val

$x :\to -3$

$\sqrt{x^2} = x$? NO

$\sqrt{(-3)^2} = -3$? NO

Fact: $\sqrt{x^2} = |x|$

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

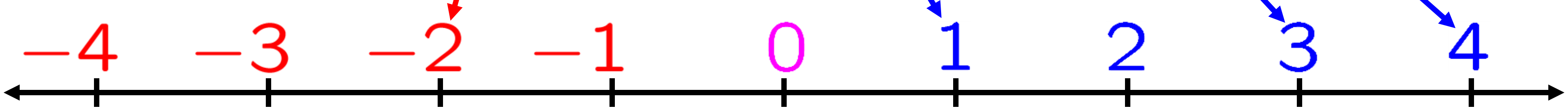
distance from 1 to 4 is: $|4 - 1|$

distance from -2 to 3 is: $|3 - (-2)|$

distance from 3 to 2 $\overbrace{\text{is:}}^{\text{NOT}}$ $|2 \overset{??}{-} 3| < 0$

distance from a to b $\overbrace{\text{is:}}^{\text{NOT}}$ $|b \underset{??}{-} a|$

$a :\longrightarrow 3$ $\quad$ $b :\longrightarrow 2$



SKILL
comp dist

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

$$|x| = |x - 0| = \text{dist}(x, 0)$$

$$|x| \leq r \quad \Leftrightarrow \quad -r \leq x \leq r$$

$$|x - a| = \text{dist}(x, a)$$

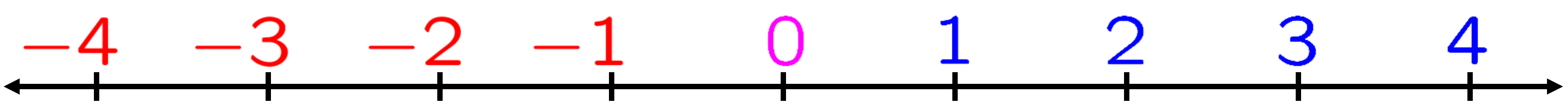
$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

$\|$

$:$

$\boxed{\text{dist}(a, b)}$



absolute value of x

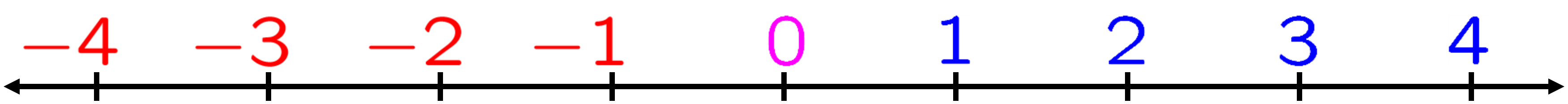
Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

strict ineq.: $|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$

$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$

distance from a to b is: $|b - a|$

$$\text{distance from } a \text{ to } b =: \boxed{\text{dist}(a, b)}$$



absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

$$|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$

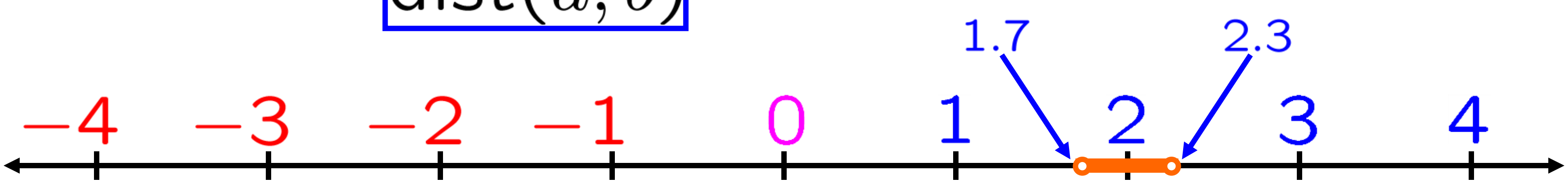
$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

$\overset{\cdot\cdot}{\|}$

$\boxed{\text{dist}(a, b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x - 2| < 0.3$. $\Leftrightarrow$ $x \in (1.7, 2.3)$

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

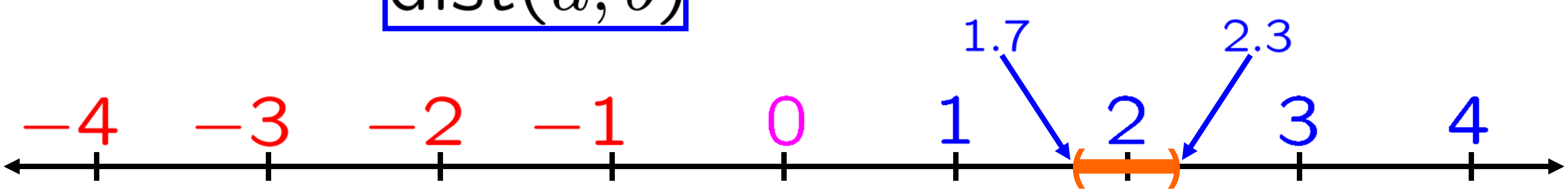
$$|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$
$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

$\shortparallel$:

$\boxed{\text{dist}(a, b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x - 2| < 0.3$. $\Leftrightarrow$ $x \in (1.7, 2.3)$

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

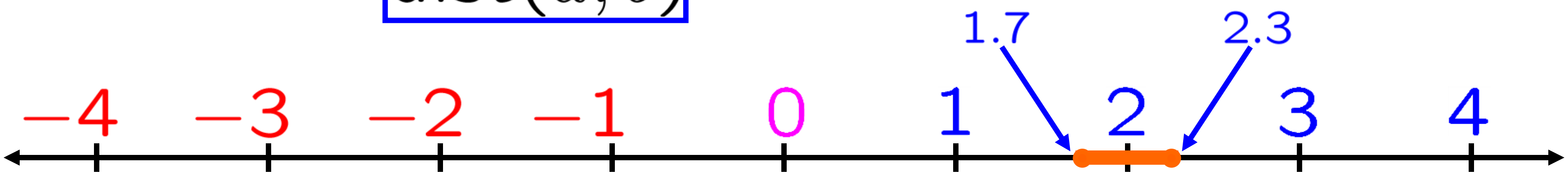
$$|x-a| < r \iff a-r < x < a+r$$
$$|x-a| \leq r \iff a-r \leq x \leq a+r$$

distance from a to b is: $|b-a|$

$=:$

$\boxed{\text{dist}(a,b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x-2| \leq 0.3$. $\iff x \in [1.7, 2.3]$

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

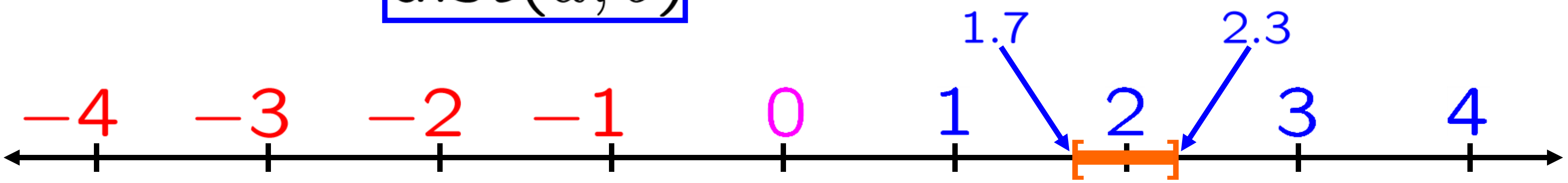
$$|x - a| < r \iff a - r < x < a + r$$
$$|x - a| \leq r \iff a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

$=:$

$\boxed{\text{dist}(a, b)}$

SKILL
gph abs val
inequality



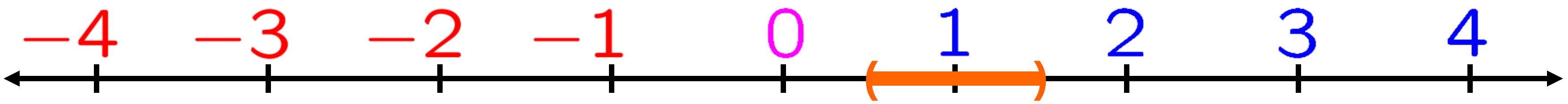
Exercise: Graph $|x - 2| \leq 0.3$. $\Leftrightarrow x \in [1.7, 2.3]$

Exercises:

Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

$$|x-a| < r \quad \Leftrightarrow \quad a-r < x < a+r$$
$$|x-a| \leq r \quad \Leftrightarrow \quad a-r \leq x \leq a+r$$

SKILL
gph nbd



0.5-neighborhood of 1

Exercises:

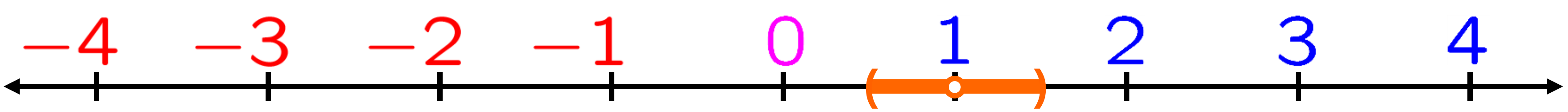
Graph $|x-1|<0.5.$ $\Leftrightarrow$ $x\in(0.5,1.5)$

Graph $0<|x-1|<0.5.$ $\Leftrightarrow$ $x\in(0.5,1.5)\boxed{\setminus}\{1\}$

set-theoretic subtraction

$$
\begin{aligned}
|x-a|<r &\quad\Leftrightarrow\quad a-r<x<a+r\\
|x-a|\le r &\quad\Leftrightarrow\quad a-r\le x\le a+r
\end{aligned}
$$

SKILL
gph nbd



punctured 0.5-neighborhood of 1

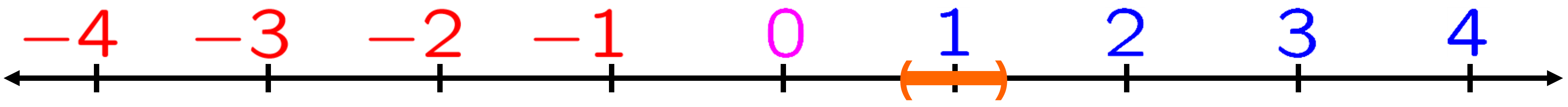
Exercises:

Graph $|x-1| < 0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x-1| < 0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5) \backslash \{1\}$

Graph $|x-1| < 0.3$. $\Leftrightarrow$ $x \in (0.7, 1.3)$

SKILL
gph nbd



0.3-neighborhood of 1

Exercises:

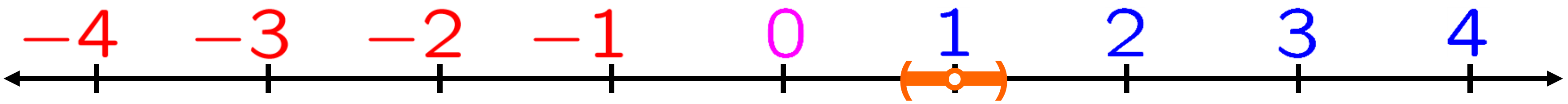
Graph $|x - 1| < 0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x - 1| < 0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x - 1| < 0.3$. $\Leftrightarrow$ $x \in (0.7, 1.3)$

Graph $0 < |x - 1| < 0.3$. $\Leftrightarrow$ $x \in (0.7, 1.3)\backslash\{1\}$

SKILL
gph nbd



punctured 0.3-neighborhood of 1

Exercises:

Graph $|x-1|<0.5.$ $\Leftrightarrow$ $x\in(0.5,1.5)$

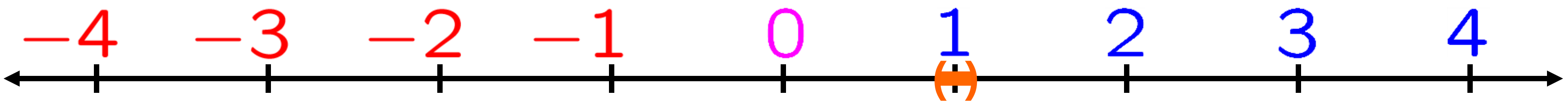
Graph $0<|x-1|<0.5.$ $\Leftrightarrow$ $x\in(0.5,1.5)\backslash\{1\}$

Graph $|x-1|<0.3.$ $\Leftrightarrow$ $x\in(0.7,1.3)$

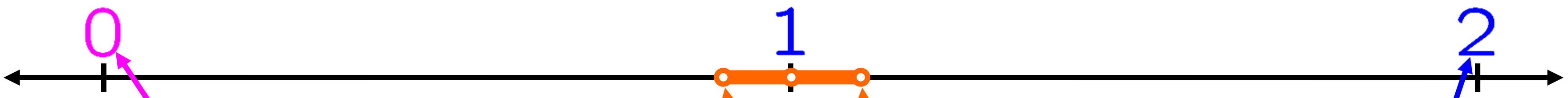
Graph $0<|x-1|<0.3.$ $\Leftrightarrow$ $x\in(0.7,1.3)\backslash\{1\}$

Graph $|x-1|<0.1.$ $\Leftrightarrow$ $x\in(0.9,1.1)$

SKILL
gph nbd



0.1-neighborhood of 1



Exercises:

Graph $|x-1|<0.5$. $\Leftrightarrow$ $x\in(0.5,1.5)$

Graph $0<|x-1|<0.5$. $\Leftrightarrow$ $x\in(0.5,1.5)\backslash\{1\}$

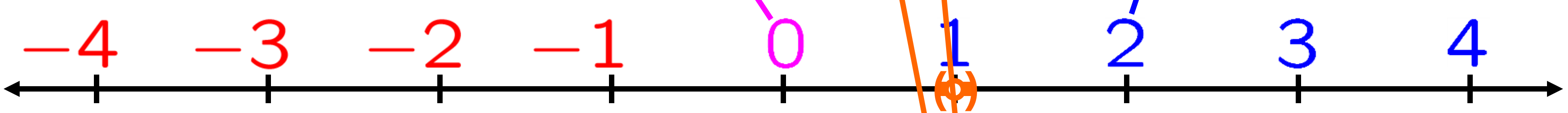
Graph $|x-1|<0.3$. $\Leftrightarrow$ $x\in(0.7,1.3)$

Graph $0<|x-1|<0.3$. $\Leftrightarrow$ $x\in(0.7,1.3)\backslash\{1\}$

Graph $|x-1|<0.1$. $\Leftrightarrow$ $x\in(0.9,1.1)$

Graph $0<|x-1|<0.1$. $\Leftrightarrow$ $x\in(0.9,1.1)\backslash\{1\}$

SKILL
gph nbd



punctured 0.1-neighborhood of 1

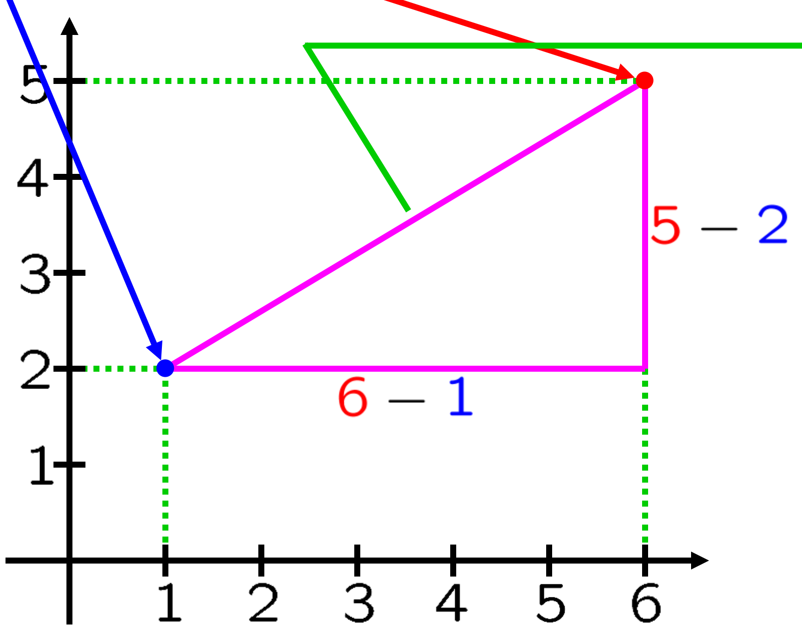
should change units, for visibility:

On the line,

$$\mathrm{dist}(a, s) = |s - a| = \sqrt{(s - a)^2}$$

In the plane,

$$\mathrm{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$



On the line,

$$\mathrm{dist}(a, s) = |s - a| = \sqrt{(s-a)^2}$$

In the plane,

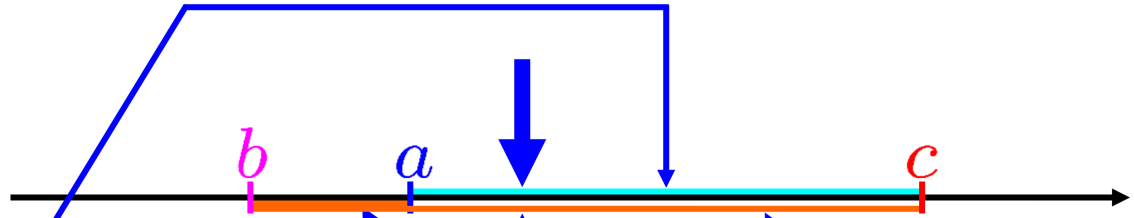
$$\mathrm{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$

$$\mathrm{dist}((a,b),(s,t)) = \sqrt{(s-a)^2 + (t-b)^2}$$

In three dimensions,

$$\mathrm{dist}((a,b,c),(s,t,u))$$

$$= \sqrt{(s-a)^2 + (t-b)^2 + (u-c)^2}$$



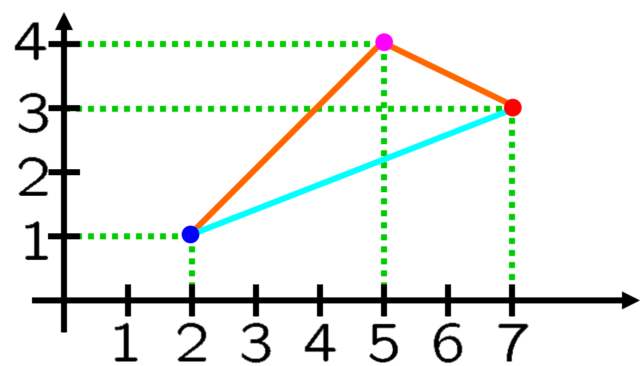
Fact (triangle inequality on the line):

$$\text{dist}(a, c) \leq [\text{dist}(a, b)] + [\text{dist}(b, c)]$$

Fact (triangle inequality on the plane):

$$\text{dist}((a, p), (c, r)) \leq [\text{dist}((a, p), (b, q))] + [\text{dist}((b, q), (c, r))]$$

e.g.: $\text{dist}((2, 1), (7, 3)) \leq [\text{dist}((2, 1), (5, 4))] + [\text{dist}((5, 4), (7, 3))]$



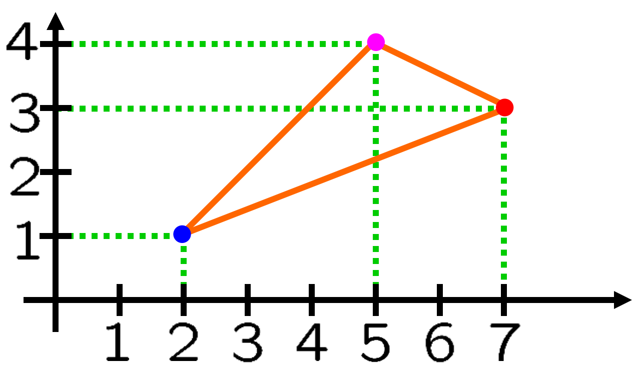
Fact (triangle inequality on the line):

$$\text{dist}(a, c) \le [\text{dist}(a, b)] + [\text{dist}(b, c)]$$

Fact (triangle inequality on the plane):

$$\text{dist}((a, p), (c, r)) \le [\text{dist}((a, p), (b, q))] + [\text{dist}((b, q), (c, r))]$$

e.g.: $\text{dist}((2, 1), (7, 3)) \le [\text{dist}((2, 1), (5, 4))] + [\text{dist}((5, 4), (7, 3))]$



Fact (triangle inequality on the line):

$$\underbrace{\mathsf{dist}(a,c)}_{|a-c|} \leq [\underbrace{\mathsf{dist}(a,b)}_{|a-b|}] + [\underbrace{\mathsf{dist}(b,c)}_{|b-c|}]$$

Fact (triangle inequality on the line):

$$|a-c| \le \underbrace{|a-b|}_{u} + \underbrace{|b-c|}_{v}$$

$$|a-c| \le |a \cdot b| + |b \cdot c|$$

Fact (triangle inequality on the line):

$$\underbrace{|a - c|}_{|u + v|} \quad \le \quad \underbrace{|a - b|}_{|u|} \quad + \quad \underbrace{|b - c|}_{|v|}$$

$$a - c \quad = \quad \overbrace{(a - \cancel{b})}^{u} \quad + \quad \overbrace{(\cancel{b} - c)}^{v}$$

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \leq |u| + |v|$$

$$|u+v| \leq |u| + |v|$$

"The absolute value of the sum
is less than or equal to
the sum of the absolute values."

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \leq |u| + |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

Intuition:

$$x \approx 3 \qquad (\text{error} < 0.002)$$

$$y \approx 5 \qquad (\text{error} < 0.007)$$

$$x+y \approx 3+5 \qquad (\text{error} < ??)$$

0.009

Pf. . .

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \quad \leq \quad |u| \quad + \quad |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$\begin{aligned}
x+y-3-5 \quad &= (x-3)+(y-5) \\
|\underbrace{(x+y)-(3+5)}| &= |(x-3)+(y-5)| \\
&\leq |x-3|+|y-5| \\
&< 0.002+0.007
\end{aligned}$$

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \quad \le \quad |u| \quad + \quad |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$|(x+y)-(3+5)| < 0.002 + 0.007$$

$$|(x+y)-(3+5)|$$

Using algebra and the triangle inequality,
we have proved this from these.

$$< 0.002 + 0.007$$

“Absolute value is ‘subadditive’.”

Fact (triangle inequality on the line):

$$|u+v| \quad \leq \quad |u| \quad + \quad |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$|(x+y)-(3+5)| < 0.002 + 0.007$$

Fact (additivity of error):

$$|x-s| < \sigma \text{ and } |y-t| < \tau$$

$$\Rightarrow \quad |(x+y)-(s+t)| < \sigma + \tau$$

Exercise: Using algebra and the triangle inquality, prove this from these.

$$\left.\begin{array}{c}\text{positive}\\ \text{part of}\\ x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

e.g.: $3_+ = 3$

$7_+ = 7$

$0_+ = 0$

$(-2)_+ = 0$

$(-5)_+ = 0$

SKILL
compute x_+

positive part of x $\Big\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$

DIVIDE BY 2

$$2x_+ = \qquad |x| + x = \begin{cases} x + x \quad , & \text{if } x > 0 \\ \boxed{0 + \boxed{x}} \quad , & \text{if } x = 0 \\ \boxed{-x + x} \quad , & \text{if } x < 0 \end{cases}$$

0 0 0

$|-x| = |x|$ $\qquad -x$

$$x_+ = \frac{|x| + x}{2}$$

NEGATIVE PART

$$\boxed{x_-} := (-x)_+ = \frac{|x| - x}{2}$$

$$\left.\begin{array}{c}\text{positive}\\ \text{part of}\\ x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

DIVIDE BY 2

$$|x| - x = \begin{cases} \boxed{x - x}, & \text{if } x > 0 \\ \boxed{0 - \boxed{x}}, & \text{if } x = 0 \\ -x - x, & \text{if } x < 0 \end{cases}$$

0 0 0

$x_+ = \dfrac{\lvert x\rvert + x}{2}$	NEGATIVE PART $\boxed{x_-} := (-x)_+ = \dfrac{\lvert x\rvert - x}{2}$

$$\left.\begin{array}{c}\text{positive}\\ \text{part of}\\ x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

DIVIDE BY 2

$$|x| - x = \begin{cases} \boxed{x - x}, & \text{if } x > 0 \\ \boxed{0 - \boxed{x}}, & \text{if } x = 0 \\ -x - x, & \text{if } x < 0 \end{cases}$$

0 0 0

DIVIDE BY 2

$$x_+ = \frac{|x| + x}{2}$$

NEGATIVE PART

$$\boxed{x_-} := (-x)_+ = \frac{|x| - x}{2}$$

$$x_- = \begin{cases} 0, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{c}\text{positive}\\\text{part of}\\x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{c}\text{negative}\\\text{part of}\\x\end{array}\right\} = x_- = \begin{cases} 0, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$x_+ = \frac{|x| + x}{2}$$

NEGATIVE PART

$$\boxed{x_-} := (-x)_+ = \frac{|x| - x}{2}$$

$$x_- = \begin{cases} 0, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{c}\text{positive}\\ \text{part of}\\ x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{c}\text{negative}\\ \text{part of}\\ x\end{array}\right\} = x_- = \begin{cases} 0, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$x_+ = \frac{|x| + x}{2}$$

NEGATIVE PART

$$\boxed{x_-} := (-x)_+ = \frac{|x| - x}{2}$$

≥ 0

e.g.: $3_- = 0$

$7_- = 0$

$0_- = 0$

$(-2)_- = 2$

$(-5)_- = 5$

SKILL
compute x_-

$$\left.\begin{array}{c}\text{positive}\\ \text{part of}\\ x\end{array}\right\} = \boxed{x_+} := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\left.\begin{array}{c}\text{negative}\\ \text{part of}\\ x\end{array}\right\} = x_- = \begin{cases} 0, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$x_+ = \frac{|x| + x}{2}$$

NEGATIVE PART

$$\boxed{x_-} := (-x)_+ = \frac{|x| - x}{2}$$

absolute value equation $\rightarrow$ $\boxed{x_+ + x_- = |x|}$

reproducing equation $\rightarrow$ $\boxed{x_+ - x_- = x}$

STOP