Financial Mathematics

One variable differential calculus review

$$f(x) = x^2 \qquad \frac{d}{dx}[f(x)] = 2x = f'(x)$$

$$f(t) = t^2 \qquad \frac{d}{dt}[f(t)] = 2t = f'(t)$$

$$f = (\bullet)^2 \qquad f' = 2(\bullet)$$

$$f'(t) = 2t$$

ALLOWABLE NOTATION:

$$\frac{d}{dx}[f(x)] \qquad \frac{d}{dt}[f(t)] \qquad f'$$

NOT ALLOWED:

$$\dot{f}$$

$$[f(u)]' \qquad [f(t)]' \qquad \frac{df}{dx}$$

I should ask your permission. . .

USUALLY AVOIDED:

$$[f(x)]' \qquad [f(u)]' \qquad [f(t)]^{\bullet} := \frac{d}{dt}[f(t)]$$

ALLOWABLE NOTATION:

$u := \sin x$ $\qquad\qquad \sin' = \cos$

$$\frac{du}{dx} = \frac{d}{dx}u = \frac{d}{dx}\sin x = \cos x$$

$v := t^5$

$$\frac{dv}{dt} = \frac{d}{dt}v = \frac{d}{dt}t^5 = 5t^4$$

USUALLY AVOIDED:

$u := \sin x$

$u' = (\sin x)' = \cos x$

directly above

$v := t^5$

$\dot{v} = (t^5)^{\bullet} = 5t^4$

$$f(x) = \cos(x) \qquad \frac{d}{dx}[f(x)] = -\sin(x) = f'(x)$$

$$f(t) = \cos(t) \qquad \frac{d}{dt}[f(t)] = -\sin(t) = f'(t)$$

$$f = \cos \qquad f' = -\sin$$

plug in x and t

$$f(x) = \ln(x) \qquad \frac{d}{dx}[f(x)] = \frac{1}{x} = f'(x)$$

$$f(t) = \ln(t) \qquad \frac{d}{dt}[f(t)] = \frac{1}{t} = f'(t)$$

$$f = \ln \qquad f' = \frac{1}{\ }$$

plug in
x and t

$$\frac{d}{dx}[f(x)] = f'(x)$$

start with a function
make it an expression of x
differentiate w.r.t. x

differentiate the function
make it an expression of x

$$\frac{d}{dt}[f(t)] = f'(t)$$

start with a function
make it an expression of t
differentiate w.r.t. t

differentiate the function
make it an expression of t

$$x :\to t$$

Product Rule

("Differentiation by parts")

1st part — 2nd part

$$\frac{d}{dx}\Big[[f(x)]\,[g(x)]\Big] =$$

$$[f(x)]\left[\frac{d}{dx}(g(x))\right] + \left[\frac{d}{dx}(f(x))\right][g(x)]$$

the 1st part times the derivative of the 2nd part plus the derivative of the 1st part times the 2nd part

$$(fg)' \quad = \quad f\,g' + f'g$$

Product Rule
(“Differentiation by parts”)

$$\frac{d}{dt}\Big[[f(t)]\ [g(t)]\Big] =$$

$$[\,f(t)\,] \quad \left[\frac{d}{dt}\,(g(t))\right] + \left[\frac{d}{dt}\,(f(t))\right] \quad [\,g(t)\,]$$

the 1st part | times | the derivative of the 2nd part | plus | the derivative of the 1st part | times | the 2nd part

Product Rule
(“Differentiation by parts”)

$$\frac{d}{du}\left[\; u^2 \quad e^u \;\right] =$$

$$[\, u^2 \,] \quad [\, e^u \,] \quad + \quad [\, 2u \,] \quad [\, e^u \,]$$

the 1st part **times** the derivative of the 2nd part **plus** the derivative of the 1st part **times** the 2nd part

Product Rule
(“Differentiation by parts”)

Many equivalent alternatives, *e.g.*,. . .

$$\frac{d}{du}\left[\; u^2 \quad e^u \;\right] = 2u \; e^u + u^2 \; e^u$$

$$= [\; u^2 \;] \qquad [\; e^u \;] \quad + \quad [\; 2u \;] \qquad [\; e^u \;]$$

the 1st part	times	the derivative of the 2nd part	plus	the derivative of the 1st part	times	the 2nd part

Quotient Rule

derivative derivative
Low dee high less high dee low,

high → low →
$$\left[\frac{f}{g}\right]' = \frac{g\,f' - f\,g'}{g^2}$$

and underneath,
low squared'll go.

Exercise: Write out the formulas for

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \text{ and } \frac{d}{dt}\left[\frac{f(t)}{g(t)}\right]$$

Quotient Rule

Low dee high less high dee low,

e.g.:

$$\frac{d}{dt}\left[\frac{\sin t}{\cos t}\right] = \frac{[\cos t]\,[\cos t] - [\sin t]\,[-\sin t]}{\cos^2 t}$$

and underneath, low squared'll go.

$$\frac{d}{dt}[\tan t] = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

$$\tan' = \sec^2$$

$$x :\to t$$

Chain Rule

$$\frac{d}{dx}[f(g(x))] = [f'(g(x))]\left[\frac{d}{dx}(g(x))\right]$$

The derivative of
an expression
plugged into a function

Take the derivative of the function.

Plug in the expression.

Multiply by the derivative of the expression.

Chain Rule

$$\frac{d}{dt}\left[f\left(g(t)\right)\right] = \left[f'\left(g(t)\right)\right]\left[\frac{d}{dt}\left(g(t)\right)\right]$$

The derivative of
an expression
plugged into a function

Take the derivative of the function.

Plug in the expression.

Multiply by the derivative of the expression.

Chain Rule

function

expression

$$\frac{d}{dx}\left[\ln(\sin(x))\right] = \left[\frac{1}{\sin(x)}\right]\left[\frac{d}{dx}\cos(x)\right]$$

$$= \cot(x)$$

Take the derivative of the function.

Plug in the expression.

Multiply by the derivative of the expression.

This expression is itself
an expression inside a function.

Chain Rule

function function expression expression

$$\frac{d}{dx}\left[\ln\left(\tan\left(x^7\right)\right)\right] = \left[\frac{1}{\tan\left(x^7\right)}\right]\left[\sec^2\left(x^7\right)\right]\left[7x^6\right]$$

Take the derivative of the function.

Plug in the expression.

Multiply by the derivative of the expression.

Practice Problem

$$\frac{d}{dx}\;\frac{e^x \sin x}{\cos^{7}(x^3)}$$

$$\frac{\left[\cos^7(x^3)\right]\left[\qquad\qquad\right] - \left[e^x \sin x\right]\left[\qquad\qquad\right]}{\cos^{14}(x^3)}$$

$$\left[\frac{e^x \sin x}{\cos^7(x^3)}\right]_{x:\to 3}$$

Practice Problem

$$\frac{d}{dx} \frac{e^x \sin x}{\cos^7(x^3)}$$

$$\frac{[\cos^7(x^3)][e^x \sin x + e^x \cos x] - [e^x \sin x][\quad]}{\cos^{14}(x^3)}$$

Practice Problem

$$\frac{d}{dx}\ \frac{e^x \sin x}{\cos^7(x^3)} =$$

$$[\cos(x^3)]^7$$

$$7[\cos(x^3)]^6$$

$$\frac{[\cos^7(x^3)][e^x \sin x + e^x \cos x] - [e^x \sin x][7\cos^6(x^3)][-\sin(x^3)][3x^2]}{\cos^{14}(x^3)}$$

Logarithmic Derivative

$$\frac{d}{dx}[\ln(f(x))] = \left[\frac{1}{f(x)}\right]\left[\frac{d}{dx}[f(x)]\right] = \frac{f'(x)}{f(x)}$$

e.g.: $\frac{d}{dx}[\ln(x^2+3x+7)] = \frac{2x+3}{x^2+3x+7}$ ■

$[f(x)]\times$

$$[f(x)]\left[\frac{d}{dx}[\ln(f(x))]\right]=\left[\frac{1}{f(x)}\right]\left[\frac{d}{dx}[f(x)]\right]$$

Principle of Logarithmic Differentiation

$$\frac{d}{dx}[f(x)]=[f(x)]\left[\frac{d}{dx}[\ln(f(x))]\right]$$

Principle of Logarithmic Differentiation:
To compute the derivative of an expression,
multiply the expression
by its logarithmic derivative.

e.g.: $$\frac{d}{dx}[x^x] = [\, x^x \,]\left[\frac{d}{dx}[\ln(x^x)]\right] \qquad (x > 0)$$

Principle of Logarithmic Differentiation

$$\frac{d}{dx}[f(x)] = [f(x)]\underbrace{\left[\frac{d}{dx}[\ln(f(x))]\right]}_{\text{logarithmic derivative of } f(x)}$$

logarithmic
derivative
of $f(x)$

Principle of Logarithmic Differentiation:
To compute the derivative of an expression,
multiply the expression
by its logarithmic derivative.

$(x > 0)$

$$\frac{d}{dx}[x^x] = [\ x^x\ ]\left[\frac{d}{dx}[\underbrace{\ln(x^x)}_{x \cdot \ln(x)}]\right]$$

derivative of first part
times
second part

plus

first part
times
derivative of second part

expl: $\overbrace{1(\ln(x)) + \underbrace{x(1/x)}_{1}}$

$$= [\ x^x\ ][\ 1 + (\ln\ x)\ ] \blacksquare$$

EXAMPLE: Differentiate $y = \frac{x^{4/5}\sqrt{x^2+8}}{(3x+7)^5}$.

Common sol'n:

$$\frac{d}{dx} \rightarrow \ln y = \frac{4}{5}\ln x + \frac{1}{2}\ln(x^2+8) - 5\ln(3x+7)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{5}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+8} - 5\cdot\frac{3}{3x+7}$$

$$\frac{dy}{dx} = y\left(\frac{4}{5}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+8} - 5\cdot\frac{3}{3x+7}\right)$$

$$\frac{dy}{dx} = \frac{x^{4/5}\sqrt{x^2+8}}{(3x+7)^5}\left(\frac{4}{5}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+8} - 5\cdot\frac{3}{3x+7}\right)$$

My sol'n:

$$\frac{dy}{dx} = \frac{x^{4/5}\sqrt{x^2+8}}{(3x+7)^5}\left(\frac{4}{5}\cdot\frac{\square}{x} + \frac{1}{2}\cdot\frac{\square}{x^2+8} - 5\cdot\frac{\square}{3x+7}\right)$$

STOP