Financial Mathematics

Techniques of one variable integration

Integration by Substitution

$$\int [f(\psi(u))][\psi'(u)]\,du = \int f(x)\,dx$$

$\underbrace{\psi(u)}_{x}$, $\overbrace{[\psi'(u)]\,du}^{dx}$, $\underbrace{\psi'(u)}_{\text{DON'T FORGET!}}$

$$x = \psi(u)$$

$$x \leftrightarrow u$$

$$\frac{dx}{du} = \psi'(u)$$

$$dx = [\psi'(u)]\,du$$

Integration by Substitution

In problems, you're often given this, and asked come up with this,

$$\int [f(\underbrace{\psi(x)}_{u})]\overbrace{[\underbrace{\psi'(x)}_{\text{DON'T FORGET!}}]\,dx}^{du} = \int f(u)\ du$$

. . . hoping that this will be easier than this.

For example. . .

$$u = \psi(x)$$

$$\frac{du}{dx} = \psi'(x)$$

$$du = [\psi'(x)]\,dx$$

Integration by Substitution

$$\int e^{\overbrace{\sin x}^{}}\!\!\!\!\underbrace{}_{u}\;\overbrace{[\cos x]\,dx}^{du} = \int e^u\,du = e^u + C$$

$$= e^{\sin x} + C \quad \blacksquare$$

$$u = \sin x$$

$$du = [\cos x]\,dx$$

Integration by Substitution

Let f continue to be arbitrary.

$$\int_a^b [f(\psi(x))][\psi'(x)]\,dx = \int_{\psi(a)}^{\psi(b)} f(u) \quad du$$

e.g.

$$u = \psi(x) = x + 5$$

$$\frac{du}{dx} = \psi'(x) = 1$$

$$du = [\psi'(x)]\,dx = dx$$

Integration by Substitution

$$\int_a^b [f(\psi(x))][\psi'(x)]\,dx = \int_{\psi(a)}^{\psi(b)} f(u) \quad du$$

$$u = \psi(x) = x + 5$$

$$\frac{du}{dx} = \psi'(x) = 1$$

$$du = [\psi'(x)]\,dx = dx$$

Integration by Substitution

$$\int_a^b f(x+5)\, dx = \int_{\psi(a)}^{\psi(b)} f(u)\, du$$

$\psi(b) = b + 5$

$\psi(a) = a + 5$

$$u = \psi(x) = x + 5$$

$$\frac{du}{dx} = \psi'(x) = 1$$

$$du = [\psi'(x)]\, dx = dx$$

Integration by Substitution

$$\int_a^b f(x+5) \, dx = \int_{a+5}^{b+5} f(u) \, du$$

$u :\longrightarrow x$

$$u = \psi(x) = x + 5$$

$$\frac{du}{dx} = \psi'(x) = 1$$

$$du = [\psi'(x)]\, dx = dx$$

Integration by Substitution

$$\int_a^b f(x+5) \quad dx = \int_{a+5}^{b+5} f(x) \quad dx$$

Given a definite integral, if you change x to [x plus a constant]
then dx is unchanged, and the limits of integration
are DECREASED by the constant,
to *compensate* for the change of adding the constant to x.

$$\int_{a+5}^{b+5} f(x) \quad dx = \int_a^b f(x+5) \quad dx$$

$$f(a+5)$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}\,dx = \int_{7-5}^{8-5} e^{-(x^2/2)+(5^2/2)+9}\,dx$$

$x :\rightarrow x + 5$

$$= e^{(5^2/2)+9} \int_2^3 e^{-x^2/2}\,dx$$

$$f(b+5)$$

$$\int_{a+5}^{b+5} f(x)\,dx = \int_a^b f(x+5)\,dx$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}\,dx = \int_{7-5}^{8-5} e^{-(x^2/2)+(5^2/2)+9}\,dx$$

$$= e^{(5^2/2)+9} \int_2^3 e^{-x^2/2}\,dx$$

Def'n:

$$\boxed{\Phi(x)} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$$

IMPROPER
INTEGRAL

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9} dx = \int_{7-5}^{8-5} e^{-(x^2/2)+(5^2/2)+9} \, dx$$

$$= e^{(5^2/2)+9} \int_2^3 e^{-x^2/2} \quad dx$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \qquad = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

IMPROPER FTC:

$$\Phi'(x) = \left[\frac{e^{-t^2/2}}{\sqrt{2\pi}} \right]_{t:\to x}$$

Integration by Substitution

e.g.:

$$\int_{7}^{8} e^{-(x^2/2)+5x+9} dx = \int_{7-5}^{8-5} e^{-(x^2/2)+(5^2/2)+9} \, dx$$

$$= e^{(5^2/2)+9} \int_{2}^{3} e^{-x^2/2} \, dx$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

IMPROPER FTC:

$$\Phi'(x) = \left[\frac{e^{-t^2/2}}{\sqrt{2\pi}} \right]_{t:\to x} = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}dx = \int_{7-5}^{8-5} e^{-(x^2/2)+(5^2/2)+9}\, dx$$

$$= e^{(5^2/2)+9} \int_2^3 \underbrace{e^{-x^2/2}}^{\sqrt{2\pi}(\Phi(x))} dx$$

$$\overset{\text{FTC}}{=} e^{(5^2/2)+9}\left[\sqrt{2\pi}(\Phi(x))\right]_{x:\to 2}^{x:\to 3}$$

$$\Phi'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}\,dx$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}(\Phi(x))\right]_{x:\to 2}^{x:\to 3}$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}[(\Phi(3)) - (\Phi(2))]\right]$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}dx$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}(\Phi(x))\right]_{x:\to 2}^{x:\to 3}$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}[(\underbrace{\Phi(3)}_{0.99865}) - (\underbrace{\Phi(2)}_{0.97725})]\right]$$

$$= \cdots \blacksquare$$

Integration by Substitution

e.g.:

$$\int_7^8 e^{-(x^2/2)+5x+9}dx$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}(\Phi(x))\right]_{x:\to 2}^{x:\to 3}$$

$$= e^{(5^2/2)+9}\left[\sqrt{2\pi}[(\Phi(3)) - (\Phi(2))]\right]$$ ■

ACCEPTABLE ANSWER
ON AN EXAM

■

SKILL
integrate exp(quadratic)

Integration by Parts

$$\frac{d}{dx}[G(x)] = g(x)$$

new up

$$G(x)$$

old up   old down   up   up   minus   the integral of   new   new

$$\int [f(x)]\,[g(x)]\,dx = [f(x)][G(x)] - \int [f'(x)]\,[G(x)]\,dx$$

$$f'(x)$$

new down

For example. . .

$$\frac{d}{dx}[f(x)] = f'(x)$$

Integration by Parts

up up
minus the integral of
new new

$$\frac{d}{dx}[\sin x] = \cos x$$

no harder

$\sin x$ ← still just a trig function

$$\int x\,[\cos x]\,dx = x\,[\sin x] - \int 1 \quad \sin x \quad dx$$

no harder

easier

1 ← polynomial of lower degree

easier

$$\frac{d}{dx}[x] = 1$$

Integration by Parts

$$\int x\,[\cos x]\,dx = x\,[\sin x] - \int 1 \;\sin x\; dx$$

$$= x\,[\sin x] + -\cos x + C$$

Looks strange, but it's correct, as sets.

Integration by Parts

I did it My Way

Problem: $\int x^2 \cos(x)\, dx$

$\sin(x)$ $\qquad\qquad$ $-\cos(x)$

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int 2x \sin(x)\, dx$$

$2x$ $\qquad\qquad$ 2

$$= x^2 \sin(x) - \left[-2x\cos(x) - \int -2\cos(x)\, dx \right]$$

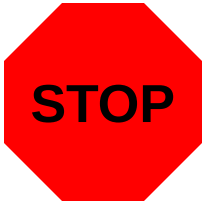
$$= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$$

"Tabular" integration

Problem: $\int x^2 \cos(x)\, dx$

	diff	antidiff
$+$	x^2	$\cos x$
$-$	$2x$	$\sin x$
$+$	2	$-\cos x$
	0	$-\sin x$

$$x^2 \sin(x)$$
$$+2x\cos(x)$$
$$-2\sin(x)$$
$$+C$$



same answer

$$x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$$