Financial Mathematics

Polynomial approximation

Jets

The n-**jet of** $f(x)$ **at** a is

$J_a^n f$ = the ordered $(n+1)$-tuple

$$(J^n f)(a) := (f(a), f'(a), f''(a), \ldots, f^{(n)}(a)).$$

e.g.:

$$f(x) = \sin x \qquad f(\pi/6) = 1/2$$
$$f'(x) = \cos x \qquad f'(\pi/6) = \sqrt{3}/2$$
$$f''(x) = -\sin x \qquad f''(\pi/6) = -1/2$$
$$f'''(x) = -\cos x \qquad f'''(\pi/6) = -\sqrt{3}/2$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(\pi/6) = 1/2$$

SKILL
compute jets

$$(J^4 f)(\pi/6) = (1/2, \sqrt{3}/2, -1/2, -\sqrt{3}/2, 1/2)$$

Jets

The **n-jet of $f(x)$ at a** is

$J_a^n f$ $=$ $(J^n f)(a)$ — the ordered $(n+1)$-tuple

$$(J^n f)(a) := (f(a), f'(a), f''(a), \ldots, f^{(n)}(a)).$$

Note: If $\tilde{f}(x) = f(-x)$, then

$$\tilde{f}'(x) = -f'(-x)$$
$$\tilde{f}''(x) = f''(-x)$$
$$\tilde{f}'''(x) = -f'''(-x)$$

etc.,

so:

$$(J^n f)(0) = (a_0, a_1, a_2, a_3, \ldots, a_n)$$
$$\Downarrow$$
$$(J^n \tilde{f})(0) = (a_0, -a_1, a_2, -a_3, \ldots, (-1)^n a_n).$$

Jets

"f and g agree to order n at 0"

$$(J^n f)(0) = (J^n g)(0)$$

$$\tilde{f}(x) = f(-x) \quad \text{and} \quad \tilde{g}(x) = g(-x)$$

$$\Downarrow$$

$$(J^n \tilde{f})(0) = (J_n \tilde{g})(0)$$

"$\tilde{f}$ and $\tilde{g}$ agree to order n at 0"

$$(J^n g)(0) = (b_0, b_1, b_2, b_3, \ldots, b_n)$$

$$\Downarrow$$

$$(J^n \tilde{g})(0) = (b_0, -b_1, b_2, -b_3, \ldots, (-1)^n b_n).$$

$$(J^n f)(0) = (a_0, a_1, a_2, a_3, \ldots, a_n)$$

$$\Downarrow$$

$$(J^n \tilde{f})(0) = (a_0, -a_1, a_2, -a_3, \ldots, (-1)^n a_n).$$

Maclaurin approximation

The **second order Maclaurin approx. of** $f(x)$ is the second degree polynomial

$$p(x) = a + bx + cx^2 \qquad \text{(degree} \le 2\text{)}$$

such that

$$f(0) = p(0),\ f'(0) = p'(0) \text{ and } f''(0) = p''(0)$$

i.e., such that

$$(J^2 f)(0) = (J^2 p)(0),$$

i.e., such that

f and p agree to order 2 at zero.

Maclaurin approximation

The **second order** **Maclaurin approx.** **of** $f(x)$ is the second degree polynomial

$$p(x) = a + bx + cx^2 \qquad \text{(degree} \le 2\text{)}$$

such that

$$f(0) = p(0), \; f'(0) = p'(0) \text{ and } f''(0) = p''(0).$$

$$p'(x) = 2cx + b \qquad p''(x) = 2c$$

$$p(0) = a \qquad p'(0) = b \qquad p''(0) = 2c$$

$$f(0) = a \qquad f'(0) = b \qquad f''(0) = 2c$$

$$p(x) = a + bx + cx^2$$

$$= [f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2}\right]x^2$$

Next: Third order. . .

Maclaurin approximation

The **third order** **Maclaurin approx.** of $f(x)$ is the third degree polynomial

$$p(x) = a + bx + cx^2 + dx^3 \qquad \text{(degree} \leq 3\text{)}$$

such that

$$f(0) = p(0),\ f'(0) = p'(0),\ \ f''(0) = p''(0)$$

$$\text{and} \quad f'''(0) = p'''(0)$$

i.e., such that

$$(J^3 f)(0) = (J^3 p)(0),$$

i.e., such that

f and p agree to order 3 at zero.

Maclaurin approximation

The **third order** **Maclaurin approx.** of $f(x)$ is the third degree polynomial

$$p(x) = a + bx + cx^2 + dx^3 \qquad \text{(degree } \le 3\text{)}$$

such that

$$f(0) = p(0),\ f'(0) = p'(0),\ f''(0) = p''(0)$$

and $f'''(0) = p'''(0)$.

$$p(x) = a + bx + cx^2 + dx^3$$

$$= [f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2}\right]x^2 + \left[\frac{f'''(0)}{6}\right]x^3$$

$$= \left[\frac{f(0)}{0!}\right]x^0 + \left[\frac{f'(0)}{1!}\right]x^1 + \left[\frac{f''(0)}{2!}\right]x^2 + \left[\frac{f'''(0)}{3!}\right]x^3$$

Next: nth order. . .

Maclaurin approximation

The n**th order Maclaurin approx. of** $f(x)$
is the polynomial of degree $\leq n$

$$p(x)$$

such that

$$f(0) = p(0), f'(0) = p'(0), \ldots, f^{(n)}(0) = p^{(n)}(0)$$

i.e., such that

$$(J^n f)(0) = (J^n p)(0),$$

i.e., such that

f and p agree to order n at zero.

Maclaurin approximation

The n**th order Maclaurin approx. of** $f(x)$ is the polynomial of degree $\leq n$

$$p(x)$$

such that

$$f(0) = p(0), f'(0) = p'(0), \ldots, f^{(n)}(0) = p^{(n)}(0).$$

$$p(x) =$$

$$= \left[\frac{f(0)}{0!}\right] x^0 + \left[\frac{f'(0)}{1!}\right] x^1 + \cdots + \left[\frac{f^{(n)}(0)}{n!}\right] x^n$$

$$= [f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2!}\right] x^2 + \cdots + \left[\frac{f^{(n)}(0)}{n!}\right] x^n$$

SKILL
compute Macl. approximations

Maclaurin expansion

The **Maclaurin expansion of** $f(x)$ is

$$[f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2!}\right]x^2 + \left[\frac{f'''(0)}{3!}\right]x^3 + \cdots$$

3rd partial sum =
2nd order Maclaurin approximation

Maclaurin expansion

The **Maclaurin expansion of** $f(x)$ is the power series whose $(n+1)$st partial sum is the nth order Maclaurin approx. of $f(x)$, for all integers $n \geq 0$.

$$[f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2!}\right]x^2 + \left[\frac{f'''(0)}{3!}\right]x^3 + \cdots$$

4th partial sum =
3rd order Maclaurin approximation

etc.

Maclaurin expansion

The **Maclaurin expansion of** $f(x)$ is the power series whose $(n+1)$st partial sum is the nth order Maclaurin approx. of $f(x)$, for all integers $n \geq 0$.

$$[f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2!}\right]x^2 + \left[\frac{f'''(0)}{3!}\right]x^3 + \cdots$$

e.g.:

e^x: $\quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\cos x$: $\quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin x$: $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\ln(1+x)$: $\quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

SKILL
compute Macl. expansions

Maclaurin expansion

The **Maclaurin expansion of** $f(x)$ is the power series whose $(n+1)$st partial sum is the nth order Maclaurin approx. of $f(x)$, for all integers $n \geq 0$.

$$[f(0)] + [f'(0)]x + \left[\frac{f''(0)}{2!}\right]x^2 + \left[\frac{f'''(0)}{3!}\right]x^3 + \cdots$$

e.g.:

$$e^x \overset{??}{=} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x \overset{??}{=} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x \overset{??}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln(1+x) \overset{??}{=} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

We will address this question soon.

Do we get equality here?

f is **decreasing** **on** I if: $\forall u, v \in I, \quad u < v \Rightarrow f(v) < f(u)$

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

f is **nonincreasing** **on** I if: $\forall u, v \in I, \quad u \leq v \Rightarrow f(v) \leq f(u)$

NONINCREASING TEST:

If $f'(x) \leq 0$, for all x in an interval I,
then f is nonincreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,
then $g(x) \leq h(x)$, for all $x \in I$.

Proof: $f := g - h$

$$f'(x) = [g'(x)] - [h'(x)] \leq 0, \text{ for all } x \in I$$

f is nonincreasing on I.

$$[g(x)] - [h(x)] = f(x) \leq f(a) = 0, \text{ for all } x \in I$$

f is **decreasing on** I if: $\forall u, v \in I, \quad u < v \Rightarrow f(v) < f(u)$

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I,
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f is nonincreasing on I.

$$[g(x)] - [h(x)] = f(x) \leq f(a) = 0, \text{ for all } x \in I$$

$$g(x) \leq h(x), \text{ for all } x \in I$$

QED

Fact: Suppose $f'(t) \leq 10$, for all $t \geq 0$.

Suppose also $f(0) = 0$.

Then $f(t) \leq 10t$, for all $t \geq 0$.

Fact: Suppose $f'(t) \leq 8t$, for all $t \geq 0$.

Suppose also $f(0) = 0$.

Then $f(t) \leq 4t^2$, for all $t \geq 0$.

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,

then $g(x) \leq h(x)$, for all $x \in I$.

Proof: $f := g - h$

$$f'(x) = [g'(x)] - [h'(x)] \leq 0, \text{ for all } x \in I$$

f is nonincreasing on I.

$$[g(x)] - [h(x)] = f(x) \leq f(a) = 0, \text{ for all } x \in I$$

$$g(x) \leq h(x), \text{ for all } x \in I$$

QED

Fact: Suppose $f'(t) \leq 10$, for all $t \geq 0$.

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Fact: Suppose $f'(t) \leq 8t$, for all $t \geq 0$.

Suppose also $f(0) = 0$.

Then $f(t) \leq 4t^2$, for all $t \geq 0$.

Fact: Suppose $f''(t) \leq 8t^3$, for all $t \geq 0$.

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,

then $g(x) \leq h(x)$, for all $x \in I$.

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,

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Fact: Suppose $f'(t) \leq 10$, for all $t \geq 0$.

Suppose also $f(0) = 0$.

Then $f(t) \leq 10t$, for all $t \geq 0$.

Fact: Suppose $f'(t) \leq 8t$, for all $t \geq 0$.

Suppose also $f(0) = 0$.

Then $f(t) \leq 4t^2$, for all $t \geq 0$.

Fact: Suppose $f''(t) \leq 8t^3$, for all $t \geq 0$.

Suppose also $f'(0) = 0$. Suppose also $f(0) = 0$.

Then $f'(t) \leq 2t^4$, for all $t \geq 0$,

and $f(t) \leq 2t^5/5$, for all $t \geq 0$.

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,

then $g(x) \leq h(x)$, for all $x \in I$.

Question: A car drives along a road, starting at mile marker 0, with velocity ≤ 10 mph. Max distance traveled in 1 hr?

Answer: Let $f(t)$ be position at time t.

$f(0) = 0.$

$f'(t) \leq 10$, for all $t \geq 0$.

$f(t) \leq 10t$, for all $t \geq 0$.

$f(1) \leq 10.$ ■

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,
then $g(x) \leq h(x)$, for all $x \in I$.

Question: A car drives along a road,
starting at mile marker 0,
starting at velocity 0,
with acceleration $\leq$ 5 mphph.
Max distance traveled in 1 hr?

Answer: Let $f(t)$ be position at time t.

$f(0) = 0$ and $f'(0) = 0$.

$f''(t) \leq 5$, for all $t \geq 0$.

$f'(t) \leq 5t$, for all $t \geq 0$.

$f(t) \leq 5t^2/2$, for all $t \geq 0$.

$f(1) \leq 5/2$. ■

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,
then $g(x) \leq h(x)$, for all $x \in I$.

Question: A train travels along tracks,
starting at foot marker 0,
starting at velocity 0,
starting at acceleration 0,
with jerk ≤ 7 fpspsps.
Max distance traveled in 10 secs?

Answer: Let $f(t)$ be position at time t.
$f(0) = 0$ and $f'(0) = 0$ and $f''(0) = 0$.
$f'''(t) \leq 7$, for all $t \geq 0$.
$f''(t) \leq 7t$, for all $t \geq 0$.
$f'(t) \leq 7t^2/2$, for all $t \geq 0$.
$f(t) \leq 7t^3/6$, for all $t \geq 0$.
$f(10) \leq 7000/6$. ■

ANTIDIFF. OF INEQUALITIES: I an interval, $a = \min I$

If $g(a) = h(a)$ and if $g'(x) \leq h'(x)$, for all $x \in I$,
then $g(x) \leq h(x)$, for all $x \in I$.

Fact: $p :=$ the 3rd order Maclaurin approximation of g.

Assume, for all $x \in [0,5]$, that $|g^{(4)}(x)| \leq 8$.

Then $|[g(5)] - [p(5)]| \leq 8 \cdot 5^4/(4!)$.

Proof: $f := g - p$ $\qquad p^{(4)} = 0$

$f^{(4)} = g^{(4)} - p^{(4)}$

Fact: $p :=$ the 3rd order Maclaurin approximation of g.

Assume, for all $x \in [0,5]$, that $|g^{(4)}(x)| \leq 8$.

Then $|[g(5)] - [p(5)]| \leq 8 \cdot 5^4/(4!)$.

Proof: $f := g - p$ $\qquad p^{(4)} = 0$

$f^{(4)} = g^{(4)} - p^{(4)}$ $\quad f(0) = f'(0) = f''(0) = f'''(0) = 0$

$g(0) = p(0), \; g'(0) = p'(0), \; g''(0) = p''(0), \; g'''(0) = p'''(0)$

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0, \quad f'''(0) = 0$

Fact: $p :=$ the 3rd order Maclaurin approximation of g.

Assume, for all $x \in [0,5]$, that $|g^{(4)}(x)| \le 8$.

Then $|[g(5)] - [p(5)]| \le 8 \cdot 5^4/(4!)$.

$|a| \le r \Leftrightarrow -r \le a \le r$

Proof: $f := g - p$

$$f^{(4)} = g^{(4)} \qquad f(0) = f'(0) = f''(0) = f'''(0) = 0$$

$\forall x \in [0,5]$,

$$-8 \le g^{(4)}(x) \le 8$$
$$-8 \le f^{(4)}(x) \le 8$$
$$-8x \le f'''(x) \le 8x$$
$$-8x^2/2 \le f''(x) \le 8x^2/2$$
$$-8x^3/(3!) \le f'(x) \le 8x^3/(3!)$$

$x :\to 5$

$$-8x^4/(4!) \le f(x) \le 8x^4/(4!)$$

$$-8 \cdot 5^4/(4!) \le f(5) \le 8 \cdot 5^4/(4!)$$

$-r \le a \le r \Leftrightarrow |a| \le r$

$$|[g(5)] - [p(5)]| = |f(5)| \le 8 \cdot 5^4/(4!)$$ QED

Fact: $p :=$ the 3rd order Maclaurin approximation of g.

Assume, for all $x \in [0, 5]$, that $|g^{(4)}(x)| \leq 8$.

Then $|[g(5)] - [p(5)]| \leq 8 \cdot 5^4/(4!)$.

Fact: Let $a \geq 0$, $M \geq 0$ and let $n \geq 1$ be an integer.

$p :=$ the $(n-1)$st order Maclaurin approx. of g

Assume, for all $x \in [0, a]$, that $|g^{(n)}(x)| \leq M$.

Then $|[g(a)] - [p(a)]| \leq M[a^n/(n!)]$.

Fact: $p :=$ the 3rd order Maclaurin approximation of g.
Assume, for all $x \in [0,5]$, that $|g^{(4)}(x)| \leq 8$.
Then $|[g(5)] - [p(5)]| \leq 8 \cdot 5^4/(4!)$.

Fact: Let $a \geq 0$, $M \geq 0$ and let $n \geq 1$ be an integer.
$p :=$ the $(n-1)$st order Maclaurin approx. of g
Assume, for all $x \in [0,a]$, that $|g^{(n)}(x)| \leq M$.
Then $|[g(a)] - [p(a)]| \leq M[a^n/(n!)]$. $M :\longrightarrow M_n$

Fact: Assume that g is infinitely diff. at x, $\forall x \in [0,a]$.
$q :=$ the Maclaurin expansion of g
For all integers $n \geq 0$, let $M_n := \max_{[0,a]} |g^{(n)}|$.
Assume that $M_n[a^n/(n!)] \to 0$, as $n \to \infty$.
Then $g(a) = q(a)$.

Fact: $p :=$ the 3rd order Maclaurin approximation of g.

Assume, for all $x \in [0,5]$, that $|g^{(4)}(x)| \leq 8$.

Then $|[g(5)] - [p(5)]| \leq 8 \cdot 5^4/(4!)$.

Fact: Let $a \geq 0$, $M \geq 0$ and let $n \geq 1$ be an integer.

$p :=$ the $(n-1)$st order Maclaurin approx. of g

Assume, for all $x \in [0,a]$, that $|g^{(n)}(x)| \leq M$.

Then $|[g(a)] - [p(a)]| \leq M[a^n/(n!)]$. $M :\to M_n, p :\to p_n$

Fact: Assume that g is infinitely diff. at x, $\forall x \in [0,a]$.

$q :=$ the Maclaurin expansion of g

For all integers $n \geq 0$, let $M_n := \max_{[0,a]} |g^{(n)}|$.

Assume that $M_n[a^n/(n!)] \to 0$, as $n \to \infty$.

Then $g(a) = q(a)$.

as $n \to \infty$

Pf: $p_n :=$ the $(n-1)$st order Macl. approx. of g.

$$|[g(a)] - [p_n(a)]| \leq M_n[a^n/(n!)] \to 0$$

$$q(a) = \lim_{n\to\infty} p_n(a) = g(a)$$

QED

e.g.: $g(x) = e^x$, $a = 9$, $q :=$ the Macl. expansion of g

$$g^{(n)}(x) = e^x$$

$e^x > 0$ e^x is incr. in x

$$M_n := \max_{[0,9]} |g^{(n)}| = \max_{0 \le x \le 9} |e^x| = \max_{0 \le x \le 9} e^x = e^9$$

$$M_n[a^n/(n!)] = e^9[9^n/(n!)] \to 0$$

$n \to \infty$ $\to 0$

$$e^9 = g(9) = q(9)$$

$$\frac{9^{10000}}{10000!} = \frac{9}{1} \cdot \frac{9}{2} \cdots \frac{9}{10000} \approx 0$$

Fact: Assume that g is infinitely diff. at x, $\forall x \in [0, a]$.

$q :=$ the Maclaurin expansion of g

For all integers $n \ge 0$, let $M_n := \max_{[0,a]} |g^{(n)}|$.

Assume that $M_n[a^n/(n!)] \to 0$, as $n \to \infty$.

Then $g(a) = q(a)$.

Pf: $p_n :=$ the $(n-1)$st order Macl. approx. of g.

$$|[g(a)] - [p_n(a)]| \le M_n[a^n/(n!)] \to 0$$

$$q(a) = \lim_{n\to\infty} p_n(a) = g(a) \quad \text{QED}$$

e.g.: $g(x) = e^x$, $a = 9$, $q :=$ the Macl. expansion of g

$$g^{(n)}(x) = e^x$$

$$M_n := \max_{[0,9]} |g^{(n)}| = \max_{0 \le x \le 9} |e^x| = \max_{0 \le x \le 9} e^x = e^9$$

$$M_n[a^n/(n!)] = e^9[9^n/(n!)] \to 0$$

$$e^9 = g(9) = q(9) = 1 + 9 + [9^2/(2!)] + [9^3/(3!)] + \cdots$$

$9 :\to a$

$$q(x) = 1 + x + [x^2/(2!)] + [x^3/(3!)] + \cdots$$

Fact: For all $a \ge 0$,

$$e^a = 1 + a + [a^2/(2!)] + [a^3/(3!)] + \cdots$$

e.g.: $g(x) = e^{-x}$, $a = 9$

$$e^{-9} = 1 - 9 + [9^2/(2!)] - [9^3/(3!)] + \cdots$$

$$= 1 + (-9) + [(-9)^2/(2!)] + [(-9)^3/(3!)] + \cdots$$

Fact: For all $a \ge 0$,

$$e^{-a} = 1 + (-a) + [(-a)^2/(2!)] + [(-a)^3/(3!)] + \cdots$$

Fact: For all $a \geq 0$,

$$e^a = 1 + a + [a^2/(2!)] + [a^3/(3!)] + \cdots$$

Fact: For all $a \geq 0$,

$$e^{-a} = 1+(-a)+[(-a)^2/(2!)]+[(-a)^3/(3!)]+\cdots$$

Fact: For all $x \in \mathbb{R}$,

$$e^x = 1 + x + [x^2/(2!)] + [x^3/(3!)] + \cdots$$

Fact: For all $a \geq 0$,

$$e^a = 1 + a + [a^2/(2!)] + [a^3/(3!)] + \cdots$$

Fact: For all $a \geq 0$,

$$e^{-a} = 1+(-a)+[(-a)^2/(2!)]+[(-a)^3/(3!)]+\cdots$$

Fact: For all $a \geq 0$,

$$e^a = 1 + a + [a^2/(2!)] + [a^3/(3!)] + \cdots$$

Fact: For all $a \geq 0$,

$$e^{-a} = 1 + (-a) + [(-a)^2/(2!)] + [(-a)^3/(3!)] + \cdots$$

Fact: For all $x \in \mathbb{R}$,

$$e^x = 1 + x + [x^2/(2!)] + [x^3/(3!)] + \cdots$$

$$\sin x = x - [x^3/(3!)] + [x^5/(5!)] - [x^7/(7!)] + \cdots$$

$$\cos x = 1 - [x^2/(2!)] + [x^4/(4!)] - [x^6/(6!)] + \cdots$$

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e.g.: $g(x) = \ln(1 + x)$, $a = 0.5$

$$\ln 1.5 = (0.5) - [(0.5)^2/2] + [(0.5)^3/3] - [(0.5)^4/4] + \cdots$$

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e.g.: $g(x) = \ln(1 - x)$, $a = 0.5$

$$\ln 0.5 = -(0.5) - [(0.5)^2/2] - [(0.5)^3/3] - [(0.5)^4/4] - \cdots$$

Fact: For all $x \in (-1, 1]$,

$$\ln(1 + x) = x - [x^2/2] + [x^3/3] - [x^4/4] + \cdots$$

Th'm: Suppose that g'' is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of g

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $g(x) = [p(x)] + [\varepsilon(x)]x^2$.

Pf:

$g(0) = p(0)$

$$\varepsilon(x) := \frac{(g(x)) - (p(x))}{x^2} \qquad \frac{0}{0}$$

$g'(0) = p'(0)$

$$\lim_{x\to 0} \varepsilon(x) \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(g'(x)) - (p'(x))}{2x} \qquad \frac{0}{0}$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{(g''(x)) - (p''(x))}{2}$$

$g''(0) = p''(0)$

$$= \frac{(g''(0)) - (p''(0))}{2} = 0$$

QED

Th'm: Suppose that g'' is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of g

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $g(x) = [p(x)] + [\varepsilon(x)]x^2$.

$$2 :\to n$$

Th'm: Suppose that $g^{(n)}$ is continuous at 0.

$p :=$ the nth order Maclaurin approx. of g

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $g(x) = [p(x)] + [\varepsilon(x)]x^n$.

error

The Maclaurin error estimate

Key idea: error $\to 0$ faster than x^n.

$$g(x) = [p(x)] + [o(x^n)]$$

a function that tends to 0 faster than x^n, *i.e.*, x^n times some function that tends to 0.

STOP