# Financial Mathematics

## Some important indeterminate forms

$$\left[\frac{n(n-1)}{n^2}\right]_{n :\to 1000000} = \frac{\cancel{(1000000)}(999,999)}{\cancel{(1000000)}(1000000)}$$

$$\approx$$

$$1$$

$$\frac{n(n-1)}{n^2} \to 1, \text{ as } n \to \infty$$

$$\frac{n(n-1)(n-2)}{n^3} \to 1, \text{ as } n \to \infty$$

Problem: Invest \$1 in bank for one year.
How many dollars at end?

Assume: Annual nominal interest rate is $r$,
convertible $n$ times per year.

Solution: Start with 1.
Multiply by $1+(r/n)$
$n$ times.

Answer: $[1+(r/n)]^n$

Question: Continuous compounding?

$$\lim_{n\to\infty}[1+(r/n)]^n = ??$$

$$[a+b]^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n$$

REMEMBER

$$[1+b]^n = 1^n + n\,1^{n-1}b + \frac{n(n-1)}{2!}1^{n-2}b^2 + \cdots$$

$$\binom{n}{1} = n \qquad \binom{n}{2} = \frac{n(n-1)}{2!}$$

$$\lim_{n\to\infty}[1+(r/n)]^n = ??$$

$$[a+b]^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}\,a^{n-2}b^2 + \cdots$$

$$[1+b]^n = \underline{1^n} + n\,\underline{1^{n-1}b} + \frac{n(n-1)}{2!}\,\underline{1^{n-2}b^2} + \cdots$$

$$[1+b]^n = 1 + n\,b + \frac{n(n-1)}{2!}\,b^2 + \cdots$$

$$[1+(r/n)]^n = 1 + \cancel{n}\,\frac{r}{\cancel{n}} + \frac{n(n-1)}{2!}\underbrace{\left[\frac{r}{n}\right]^2}_{\left[\frac{r^2}{n^2}\right]} + \cdots$$

$$\lim_{n\to\infty}[1+(r/n)]^n = ??$$

$$[1+(r/n)]^n = 1 + r + \frac{n(n-1)}{n^2}\left[\frac{r^2}{2!}\right] + \cdots$$

$$\frac{n(n-1)}{2!}\left[\frac{r^2}{n^2}\right]$$

$$[1+(r/n)]^n = 1 + \cancel{n}\,\frac{r}{\cancel{n}} + \frac{n(n-1)}{2!}\underbrace{\left[\frac{r}{n}\right]^2}_{\left[\frac{r^2}{n^2}\right]} + \cdots$$

$$\lim_{n\to\infty}[1+(r/n)]^n = ??$$

$$[1 + (r/n)]^n \to e^r, \text{ as } n \to \infty$$

$$\underbrace{[1 + (r/n)]^n}_{n \to \infty} = \underline{1} + \underline{r} + \underbrace{\frac{n(n-1)}{n^2}}_{n \to \infty:\ 1} \underline{\left[\frac{r^2}{2!}\right]} + \cdots$$

$$1 + r + \frac{r^2}{2!} + \cdots = \boxed{e^r} = \exp(r)$$

$r$ = the nominal interest rate

$e^r$ = the "risk-free factor" under continuous compounding

$$\lim_{n \to \infty} [1 + (r/n)]^n = ??$$

$$[1+(r/n)]^n \to e^r, \text{ as } n \to \infty$$

$r :\to 7$

$$\left[1+\frac{7}{n}\right]^n \to e^7$$

Exercise: $$\left[1+\frac{7}{n}+\frac{1/n}{n}\right]^n \to e^7$$

Exercise: $$\left[1+\frac{7}{n}+\frac{(5+n)/(n+1)^3}{n}\right]^n \to e^7$$

$0$, as $n \to \infty$

Fact: $\forall \delta_n \to 0$, $$\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n \to e^7$$

Pf: $$\frac{(\ln(1+h)) - (\ln(1))}{h} \underset{h \to 0}{\longrightarrow} \ln'(1) = \frac{1}{1} = 1$$

$$\frac{\ln(1+h)}{h} \underset{h \to 0}{\longrightarrow} 1$$

$$h :\longrightarrow \frac{7}{n} + \frac{\delta_n}{n} \xrightarrow{n \to \infty} 0$$

$$\frac{\ln(1 + \frac{7}{n} + \frac{\delta_n}{n})}{\frac{7}{n} + \frac{\delta_n}{n}} \underset{n \to \infty}{\longrightarrow} 1$$

---

Fact: $\forall \delta_n \to 0,\quad \left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n \longrightarrow e^7$

Pf: $n \to \infty$:

$$\frac{\ln(1 + \frac{7}{n} + \frac{\delta_n}{n})}{\frac{7}{n} + \frac{\delta_n}{n}} \to 1$$

$$\cancel{n}\left[\frac{7}{\cancel{n}} + \frac{\delta_n}{\cancel{n}}\right]$$

$$\frac{\ln(1 + \frac{7}{n} + \frac{\delta_n}{n})}{\frac{7}{n} + \frac{\delta_n}{n}} \underset{n \to \infty}{\longrightarrow} 1$$

---

Fact: $\forall \delta_n \to 0, \quad \left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n \to e^7$

Pf: $n \to \infty$:

$$\frac{\ln\left(1 + \frac{7}{n} + \frac{\delta_n}{n}\right)}{\frac{7}{n} + \frac{\delta_n}{n}} \to 1$$

$$\cancel{n}\left[\frac{7}{\cancel{n}} + \frac{\delta_n}{\cancel{n}}\right] = 7 + \delta_n \to 7$$

Fact: $\forall \delta_n \to 0$, $\left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n \to e^7$

Pf: $n \to \infty$:

$$\frac{\ln\left(1 + \frac{7}{n} + \frac{\delta_n}{n}\right)}{\frac{7}{n} + \frac{\delta_n}{n}} \to 1$$

$$n\left[\frac{7}{n} + \frac{\delta_n}{n}\right] \to 7$$

MULTIPLY TOGETHER $\to 7$

Fact: $\forall \delta_n \to 0$, $\left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n \to e^7$

Pf: $n \to \infty$:

$$\frac{\ln\left(1 + \frac{7}{n} + \frac{\delta_n}{n}\right)}{\frac{7}{n} + \frac{\delta_n}{n}} \to 1$$

$$n\left[\frac{7}{n} + \frac{\delta_n}{n}\right] \to 7$$

MULTIPLY TOGETHER

$$n\left[\ln\left(1 + \frac{7}{n} + \frac{\delta_n}{n}\right)\right] \to 7$$

$$||$$

$$\ln\left(\left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n\right)$$

Fact: $\forall \delta_n \to 0$, $\left[1 + \frac{7}{n} + \frac{\delta_n}{n}\right]^n \to e^7$

Pf: $n \to \infty$: $\ln\left(\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n\right) \to 7$ } EXPONENTIATE

$$\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n \to e^7$$

$$\to 7$$

$$\ln\left(\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n\right)$$

---

Fact: $\forall \delta_n \to 0$, $\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n \to e^7$

Pf: $n \to \infty$: $\ln\left(\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n\right) \to 7$

$$\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n \to e^7 \qquad \text{QED}$$

Fact: $\forall \delta_n \to 0$, $\left[1+\frac{7}{n}+\frac{\delta_n}{n}\right]^n \to e^7$

$7 :\to x$

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1+\frac{x}{n}+\frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**

$$\left[f\left(\frac{x}{\sqrt{n}}\right)\right]^n = \left[1 - \left(\frac{x}{\sqrt{n}}\right)^2\right]^n = \left[1 + \frac{-x^2}{n}\right]^n \to e^{-x^2}$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$

---

a sequence that tends to 0 faster than $1/n$,

*i.e.*,

$1/n$ times some sequence that tends to 0

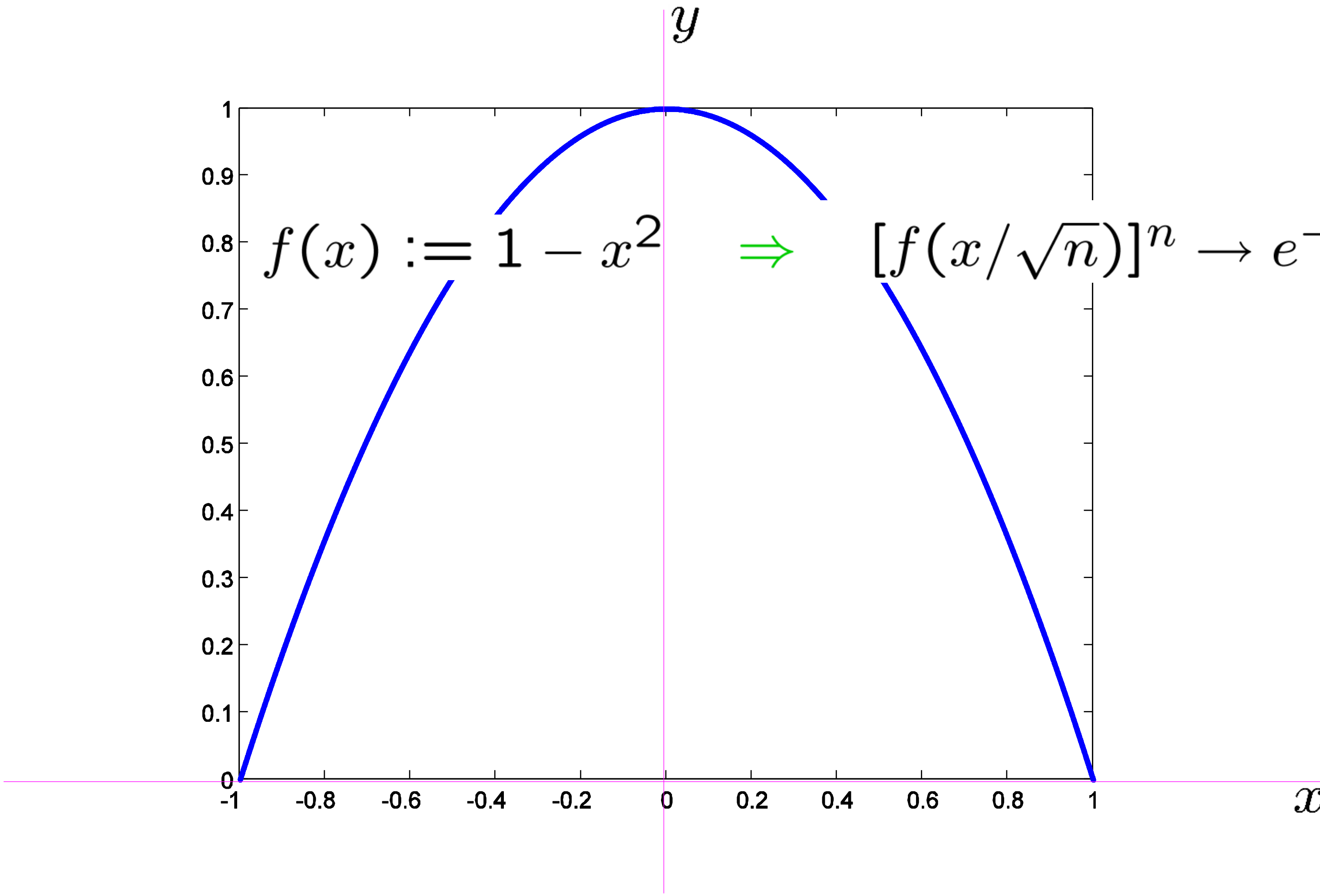
$$\left[1 + \frac{x}{n} + o\left(\frac{1}{n}\right)\right]^n \to e^x$$

---

$\delta_n :\to 0,\ x :\to -x^2$

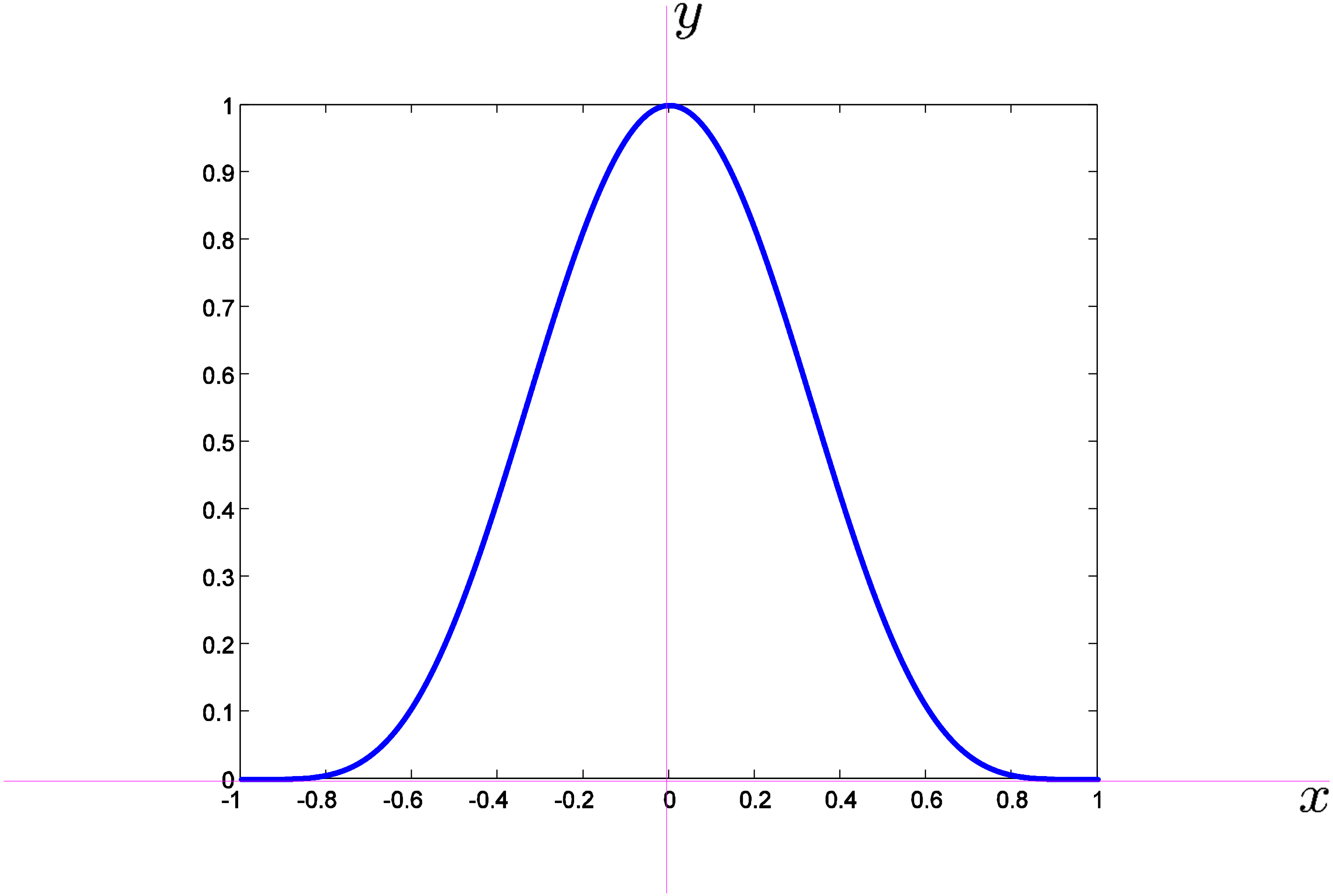
Fact: $\forall x \in \mathbb{R}, \quad \forall \delta_n \to 0, \quad \left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**



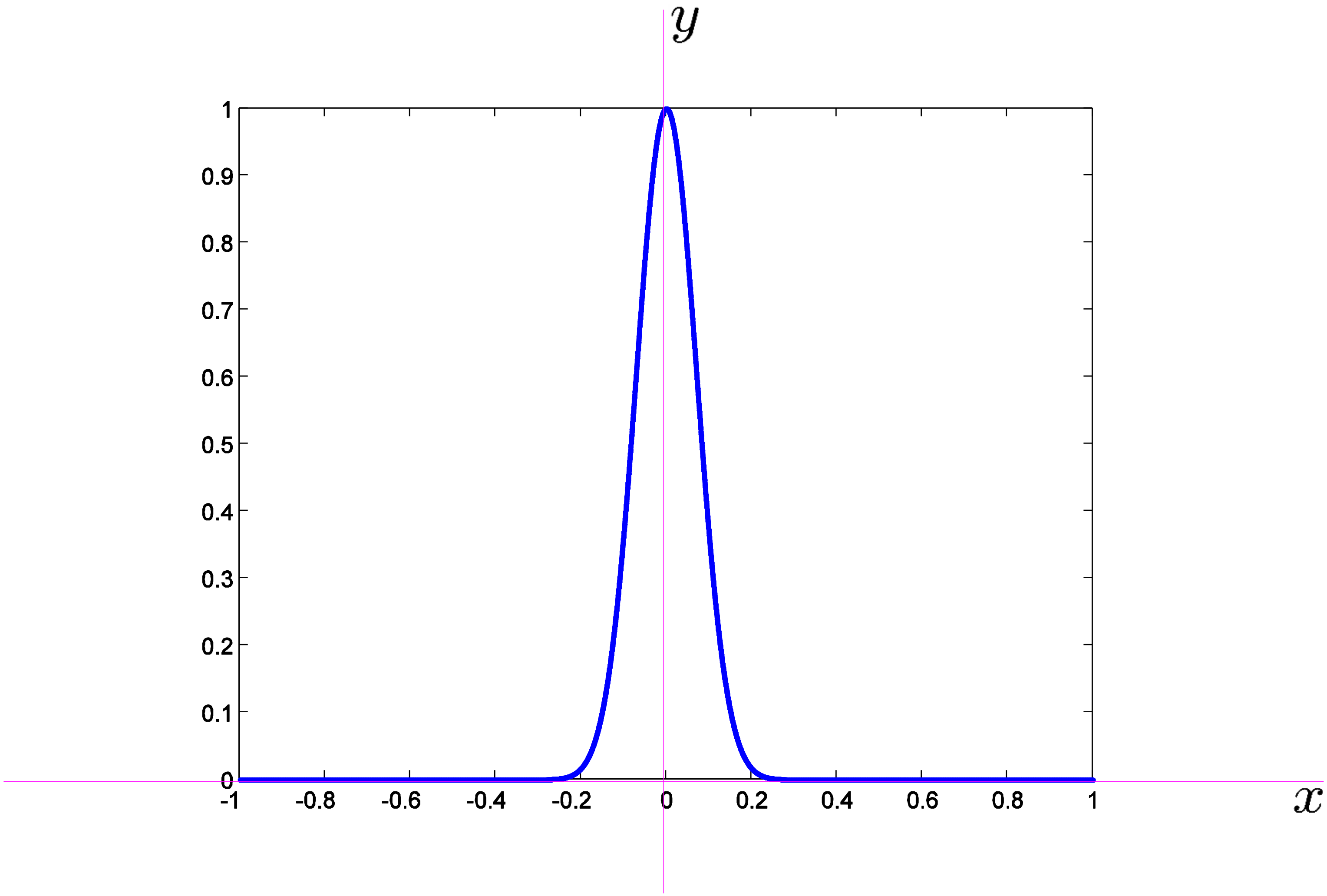

$$y = 1 - x^2$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$



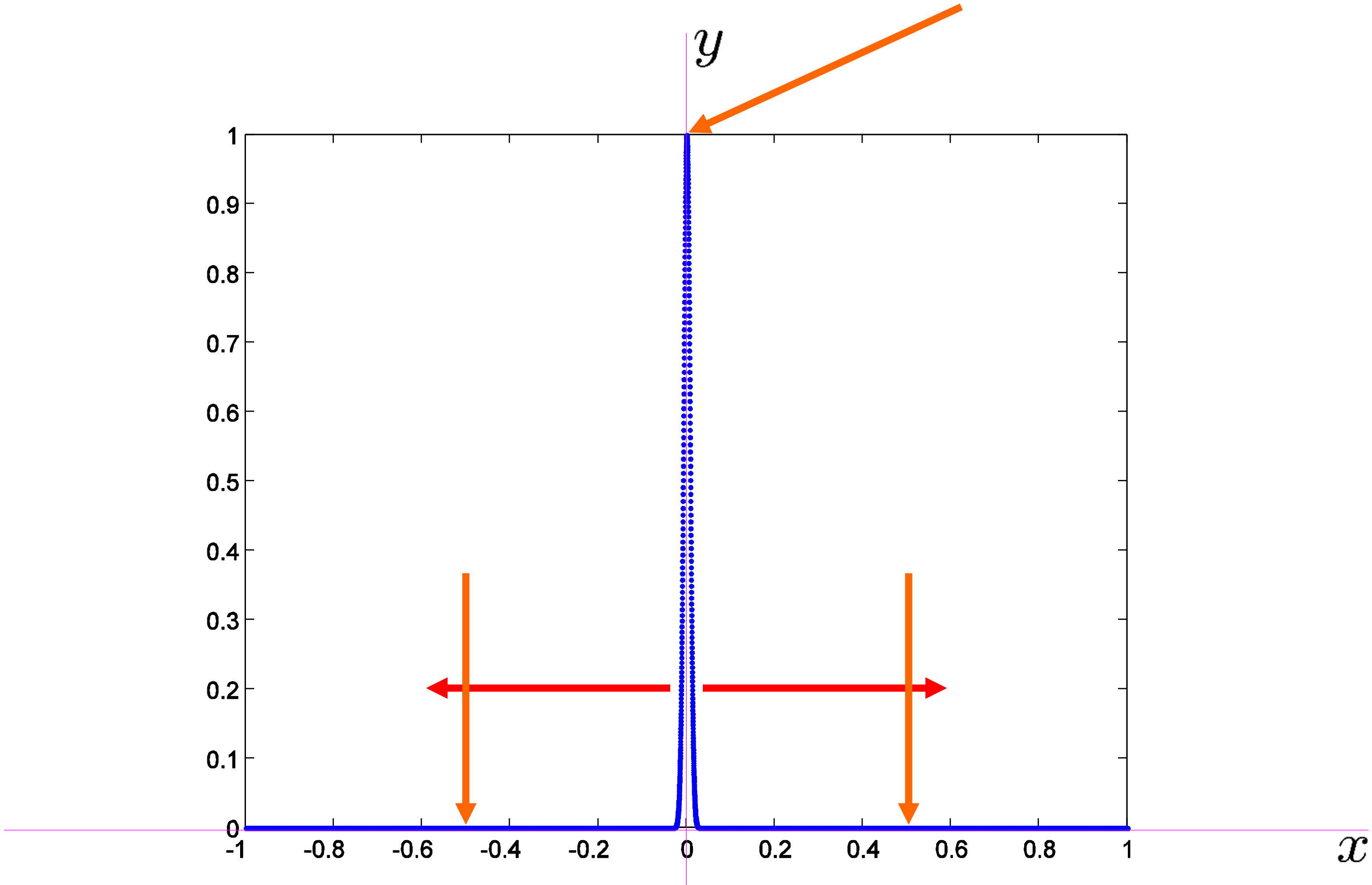

$$y = (1 - x^2)^5$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$




$$y = (1 - x^2)^{100}$$

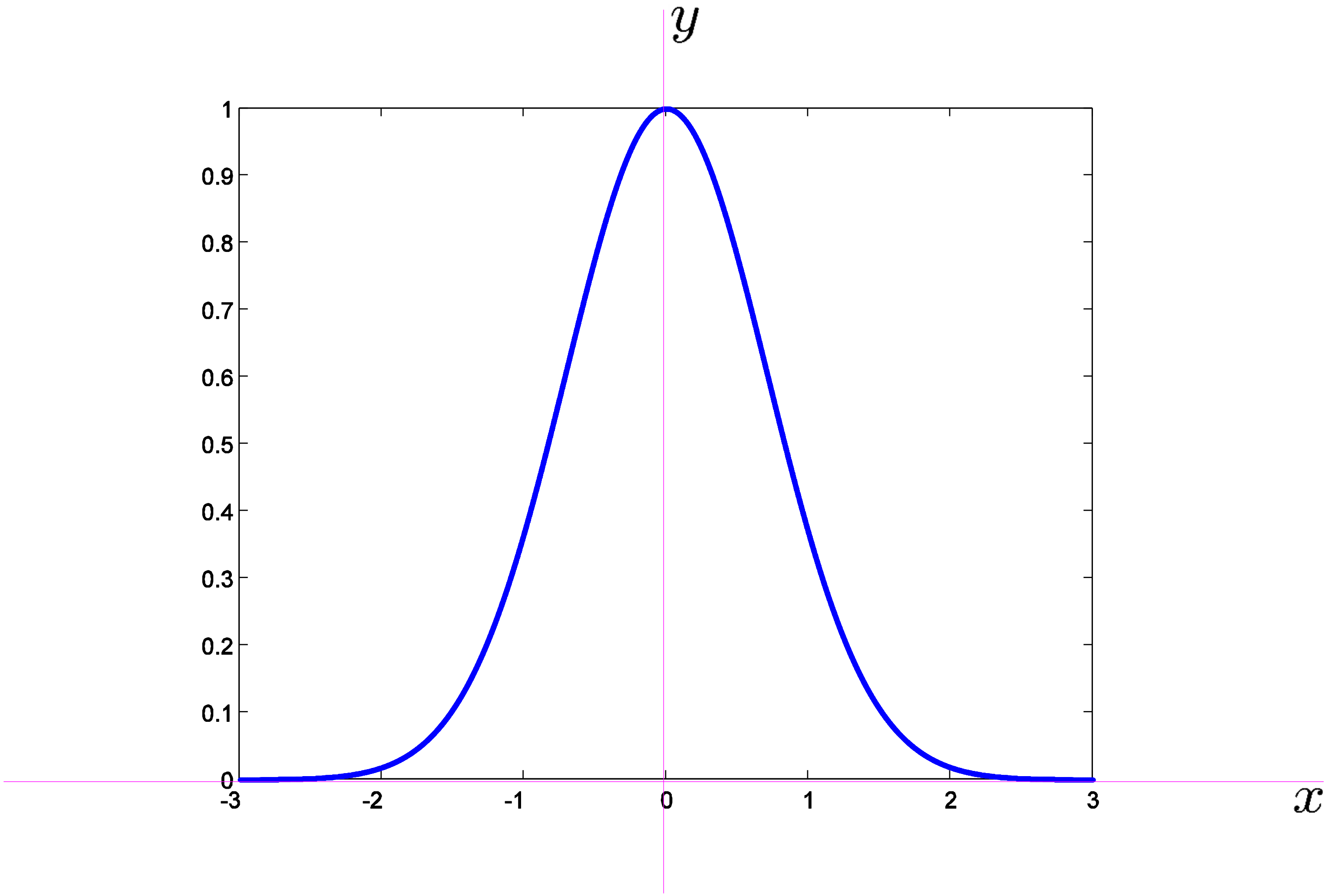
$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$




$x :\to x/\sqrt{10,000}$

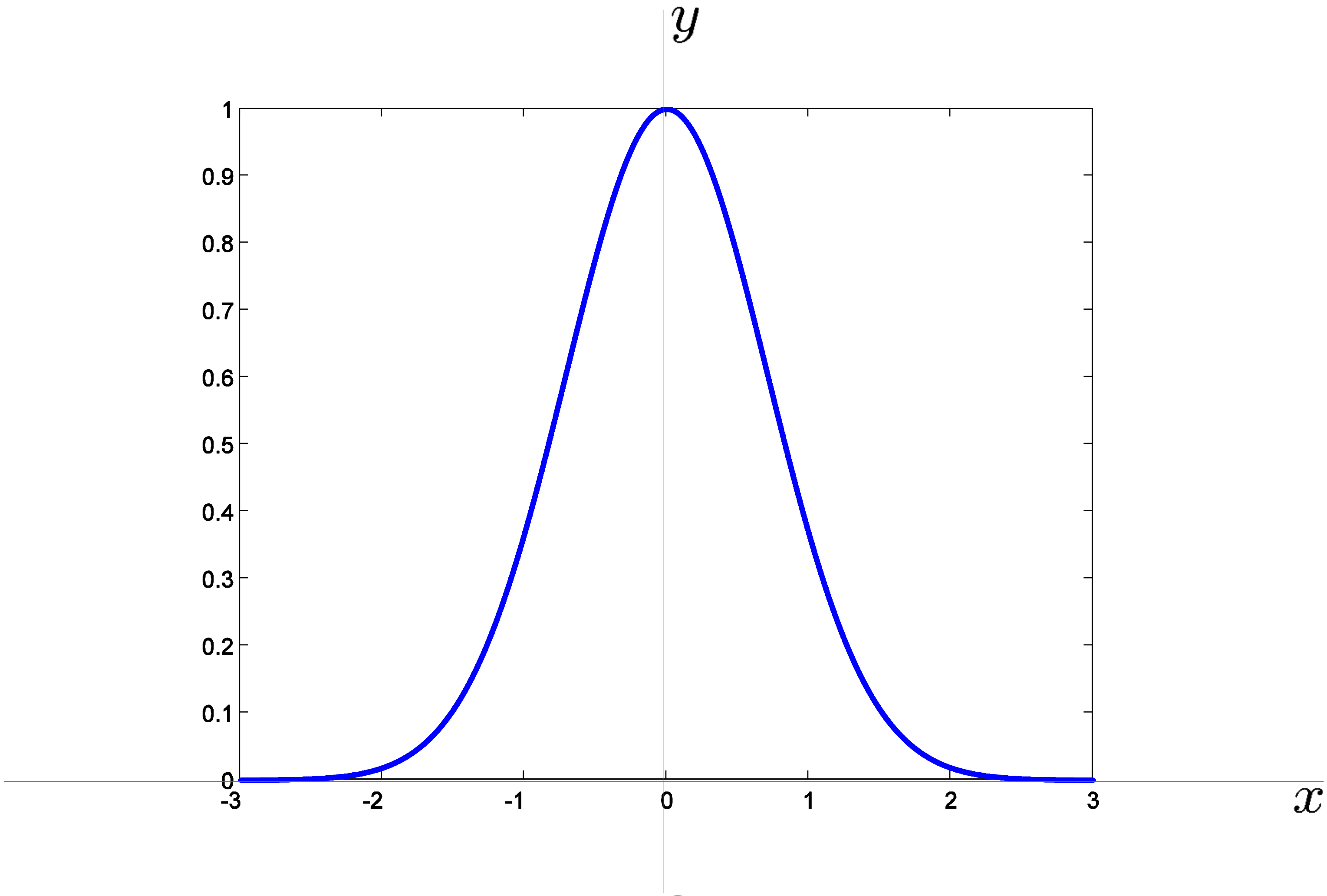
renormalization $y = (1 - x^2)^{10,000}$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$



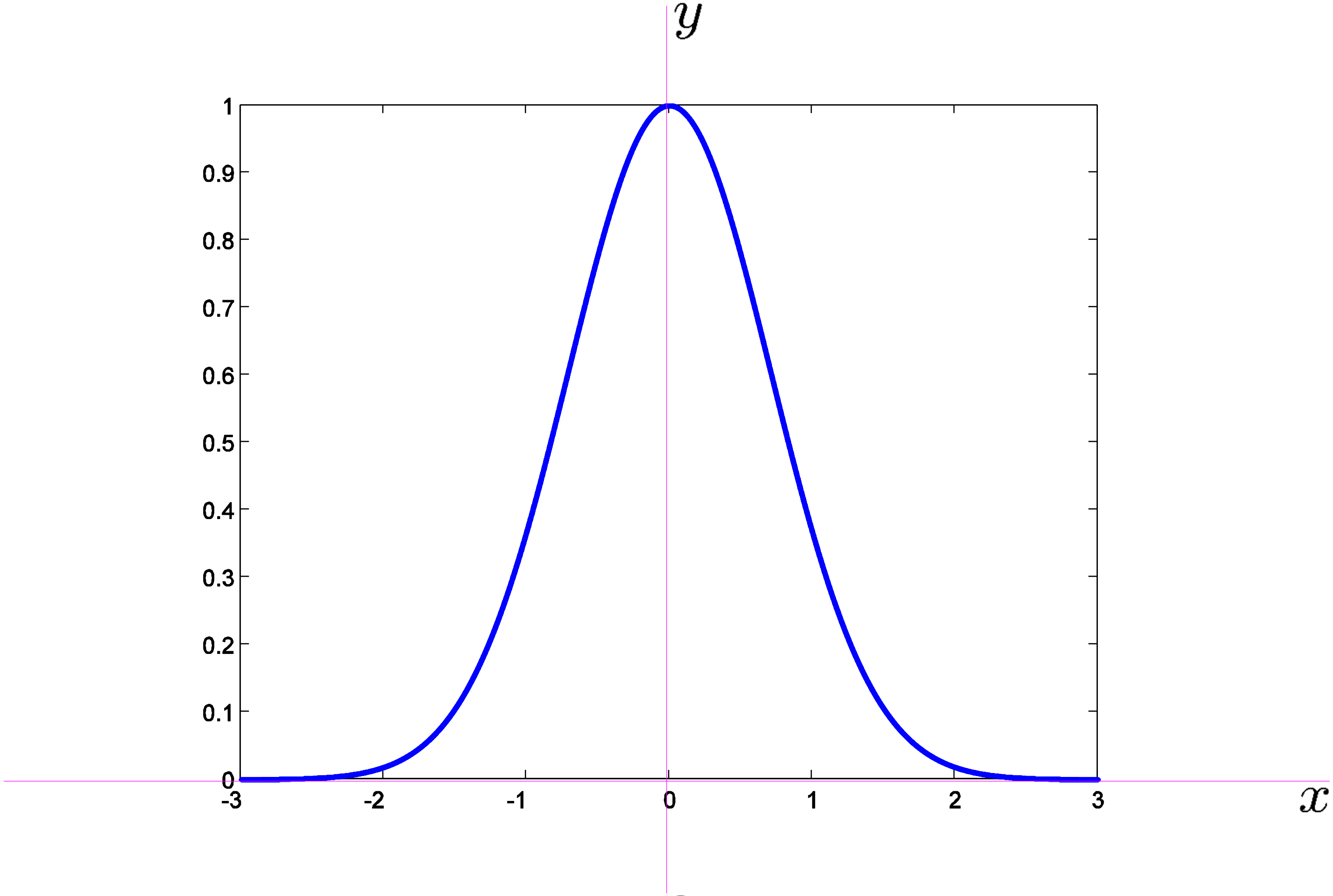

$$y = (1 - (x/\sqrt{10,000})^2)^{10,000}$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$




$$y = e^{-x^2}$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(x/\sqrt{n})]^n \to e^{-x^2}$$




$$y = e^{-x^2}$$

$$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(3/\sqrt{n})]^n \to e^{-3^2}$$

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Exercise:

$f(x) := 1 - 7x^2 + 8x^3 \quad \Rightarrow \quad [f(3/\sqrt{n})]^n \to e^{-7\cdot 3^2}$

---

Exercise:

$f(x) := 1 - 7x^2 \quad \Rightarrow \quad [f(3/\sqrt{n})]^n \to e^{-7\cdot 3^2}$

---

$f(x) := 1 - x^2 \quad \Rightarrow \quad [f(3/\sqrt{n})]^n \to e^{-3^2}$

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

To prove this, start by remembering two results. . .

---

$n :\to 2, \; g :\to f$

**The Maclaurin error estimate**

Th'm: Suppose that $g^{(n)}$ is continuous at 0.

$p :=$ the $n$th order Maclaurin approx. of $g$

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $g(x) = [p(x)] + [\varepsilon(x)]x^n$.

---

Fact: $\forall x \in \mathbb{R}, \quad \forall \delta_n \to 0, \quad \left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1-7x^2$.

Then $\lim_{n\to\infty}[f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

## The Maclaurin error estimate

Th'm: Suppose that $f'''$ is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of $f$

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $f(x) = [p(x)] + [\varepsilon(x)]x^2$.

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

## The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Pf: $p(x) := 1 - 7x^2$

$f(x) = [(p(x)] + [\varepsilon(x)]x^2$ $\qquad\qquad$ $\varepsilon(x) \to 0$, as $x \to 0$

---

Th'm: Suppose that $f'''$ is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of $f$

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $f(x) = [p(x)] + [\varepsilon(x)]x^2$.

---

Fact: $\forall x \in \mathbb{R}$, $\quad \forall \delta_n \to 0$, $\quad \left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1-7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Pf: $p(x) := 1 - 7x^2$

$$f(x) = [(p(x)] + [\varepsilon(x)]x^2 \qquad \varepsilon(x) \to 0, \text{ as } x \to 0$$

$$= [1 - 7x^2] + [\varepsilon(x)]x^2$$

---

Th'm: Suppose that $f'''$ is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of $f$

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $f(x) = [p(x)] + [\varepsilon(x)]x^2$.

---

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Pf: $p(x) := 1 - 7x^2$

$f(x)$ $\varepsilon(x) \to 0$, as $x \to 0$

$x := 3/\sqrt{n}$

$= [1 - 7x^2] + [\varepsilon(x)]x^2$

$$f(x) = [1 - 7x^2] + [\varepsilon(x)]x^2$$

$$f\left(\frac{3}{\sqrt{n}}\right) = \left[1 - \frac{7\cdot 3^2}{n}\right] + \left[\varepsilon\left(\frac{3}{\sqrt{n}}\right)\right]\left[\frac{3^2}{n}\right]$$

---

Th'm: Suppose that $f'''$ is continuous at 0.

$p :=$ the 2nd order Maclaurin approx. of $f$

Then $\exists \varepsilon(x) \to 0$, as $x \to 0$

such that $f(x) = [p(x)] + [\varepsilon(x)]x^2$.

---

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Pf: $p(x) := 1 - 7x^2$

$\varepsilon(x) \to 0$, as $x \to 0$

$$f(x) = [1 - 7x^2] + [\varepsilon(x)]x^2$$

$$f\left(\frac{3}{\sqrt{n}}\right) = \left[1 - \frac{7\cdot 3^2}{n}\right] + \left[\varepsilon\left(\frac{3}{\sqrt{n}}\right)\right]\left[\frac{3^2}{n}\right]$$

$$= 1 - \frac{7\cdot 3^2}{n} + \frac{\delta_n}{n}$$

$$\delta_n := \left[\varepsilon\left(\frac{3}{\sqrt{n}}\right)\right][3^2] \to 0, \text{ as } n \to \infty$$

---

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

Pf: $p(x) := 1 - 7x^2$

$\varepsilon(x) \to 0$, as $x \to 0$

$$f(x) = [1 - 7x^2] + [\varepsilon(x)]x^2$$

$$f\left(\frac{3}{\sqrt{n}}\right) = 1 - \frac{7\cdot 3^2}{n} + \frac{\delta_n}{n}$$

$$\left[f\left(\frac{3}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{7\cdot 3^2}{n} + \frac{\delta_n}{n}\right]^n$$

$$\delta_n := \left[\varepsilon\left(\frac{3}{\sqrt{n}}\right)\right][3^2] \to 0, \text{ as } n \to \infty$$

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1-7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

Pf: $p(x) := 1 - 7x^2$

$\varepsilon(x) \to 0$, as $x \to 0$

$$f(x) = [1 - 7x^2] + [\varepsilon(x)]x^2$$

$$f\left(\frac{3}{\sqrt{n}}\right) = 1 - \frac{7\cdot 3^2}{n} + \frac{\delta_n}{n}$$

$$\left[f\left(\frac{3}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{7\cdot 3^2}{n} + \frac{\delta_n}{n}\right]^n \to e^{-7\cdot 3^2}$$ QED

$$\delta_n := \left[\varepsilon\left(\frac{3}{\sqrt{n}}\right)\right][3^2] \to 0, \text{ as } n \to \infty$$

$x :\to -7\cdot 3^2$

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

**The exponential limit**

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$.

---

Notationally compact proof:

$$f(x) = 1 - 7x^2 + o(x^2), \text{ as } x \to 0$$

$$f(3/\sqrt{n}) = 1 - (7 \cdot 3^2/n) + \underbrace{o(3^2/n)}_{o(1/n)}, \text{ as } n \to \infty$$

$$[f(3/\sqrt{n})]^n = [1 - (7 \cdot 3^2/n) + o(1/n)]^n$$

as $n \to \infty$

---

Fact: $\forall x \in \mathbb{R}, \quad \forall \delta_n \to 0, \quad \left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$. $\quad 3 :\to x$

Notationally compact proof:

$$f(x) = 1 - 7x^2 + o(x^2), \text{ as } x \to 0$$

$$f(3/\sqrt{n}) = 1 - (7 \cdot 3^2/n) + \underbrace{o(3^2/n)}_{o(1/n)}, \text{ as } n \to \infty$$

$$[f(3/\sqrt{n})]^n = [1 - (7 \cdot 3^2/n) + o(1/n)]^n \to e^{-7\cdot 3^2},$$

as $n \to \infty$ QED

$$\left[1 + \frac{x}{n} + o\left(\frac{1}{n}\right)\right]^n \to e^x$$

Fact: $\forall x \in \mathbb{R}$, $\forall \delta_n \to 0$, $\left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(3/\sqrt{n})]^n = e^{-7\cdot 3^2}$. $3 :\longrightarrow x$

Fact: Suppose that $f'''$ is continuous at 0.

Let $f(x)$ have 2nd order Macl. approx. $1 - 7x^2$.

Then $\lim_{n\to\infty} [f(x/\sqrt{n})]^n = e^{-7x^2}$. $7 :\longrightarrow a$

Fact: Suppose that $f'''$ is continuous at 0. Let $a \in \mathbb{R}$.

Let $f(x)$ have 2nd order Macl. approx. $1 - ax^2$.

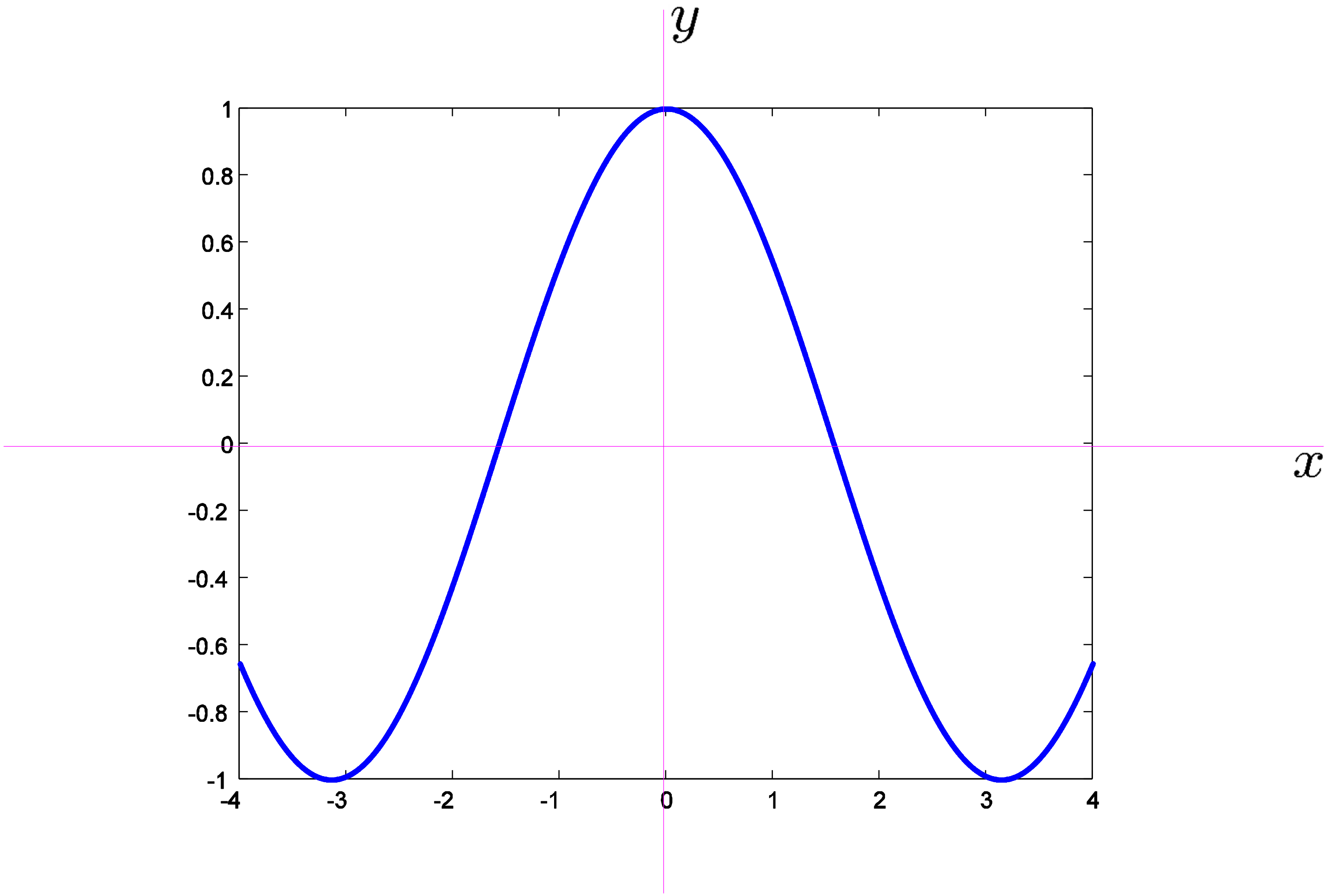
Then $\lim_{n\to\infty} [f(x/\sqrt{n})]^n = e^{-ax^2}$.

**The renormalized power limit**

e.g.: $f(x) = \cos x = 1 - \frac{x^2}{2!} + \cdots$ $\quad a = \frac{1}{2!} = \frac{1}{2}$

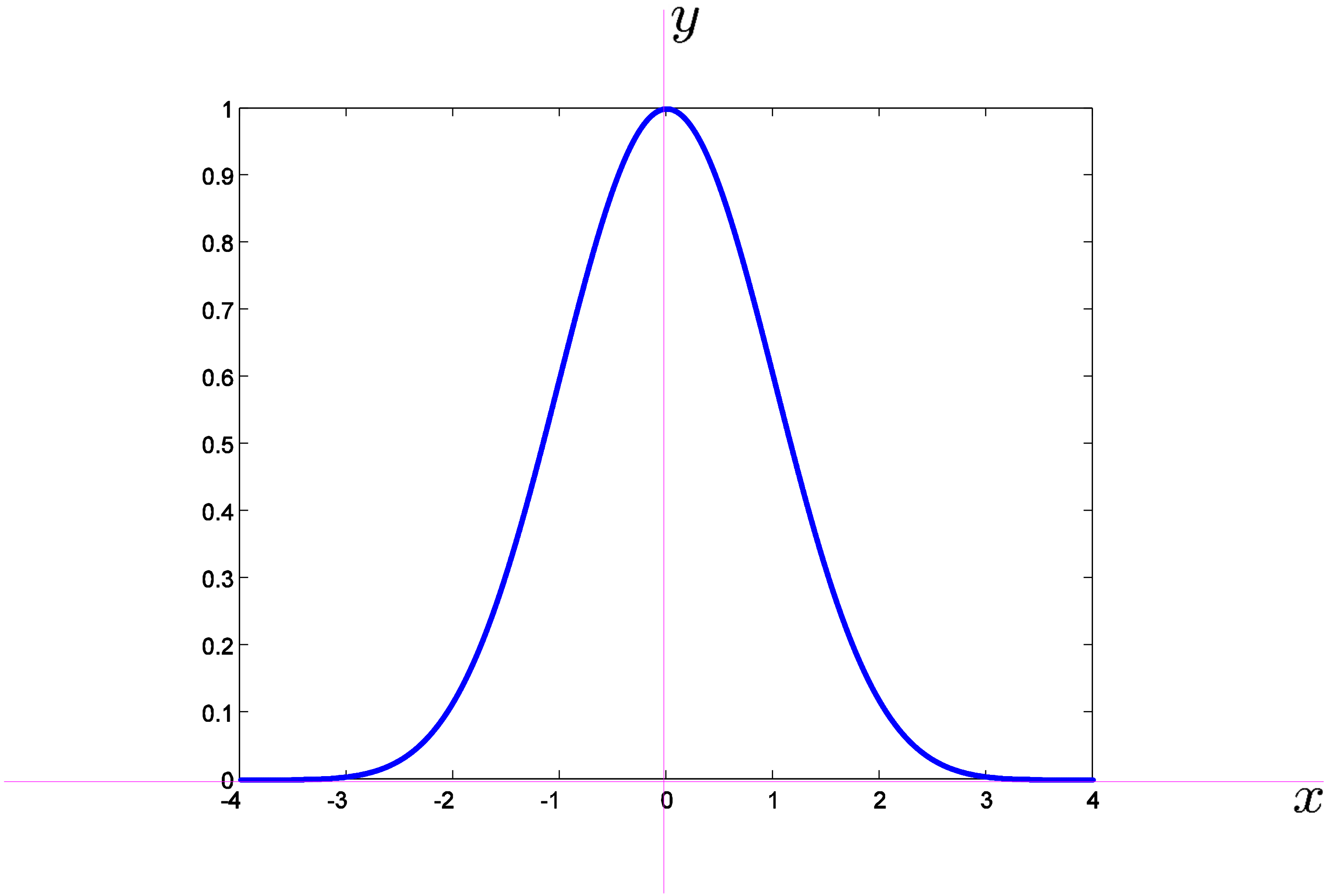
2nd order Macl. approx.

$$\lim_{n\to\infty} [\cos(x/\sqrt{n})]^n = e^{-x^2/2}$$



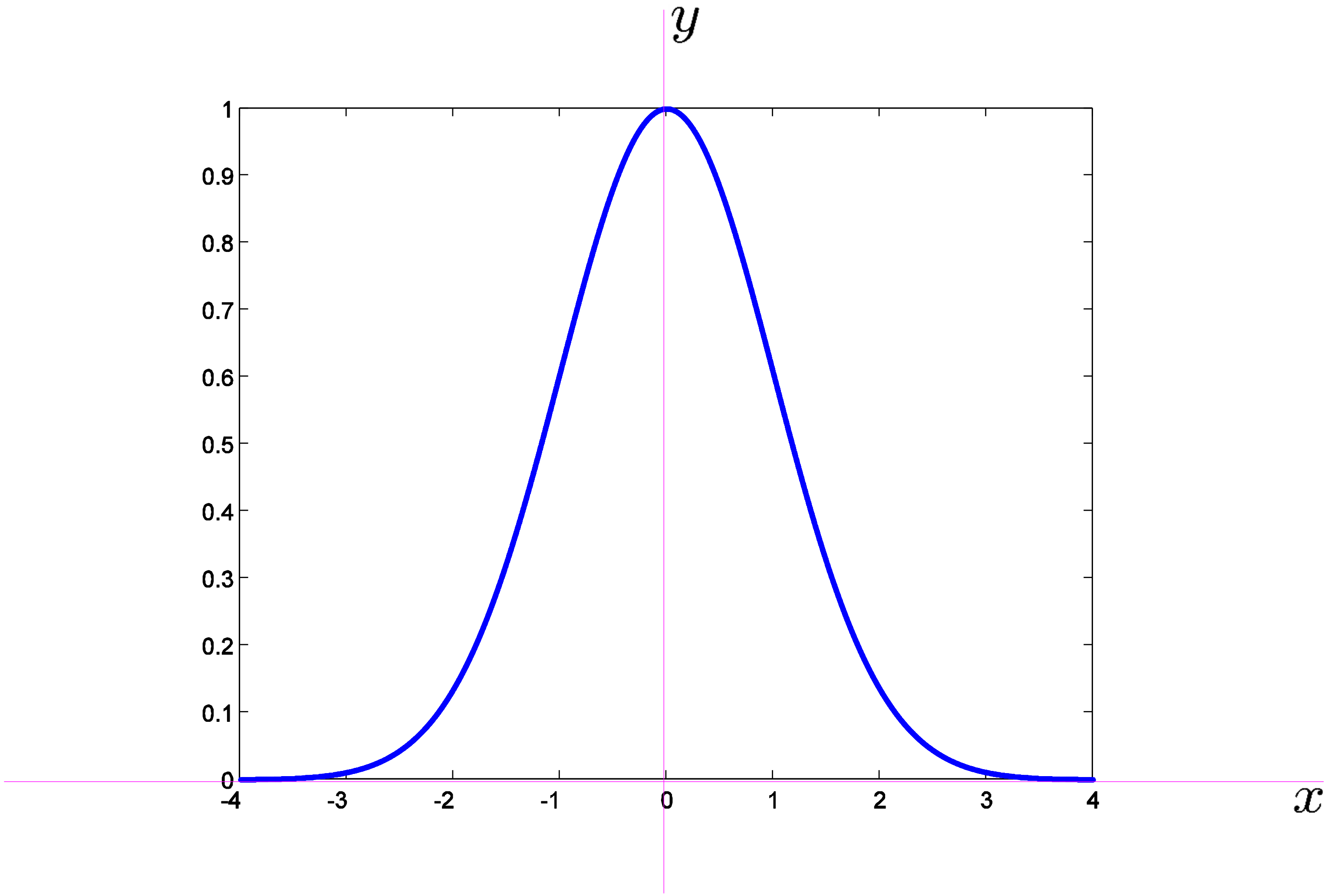

$$y = \cos x$$

$$\lim_{n \to \infty} [\cos(x/\sqrt{n})]^n = e^{-x^2/2}$$



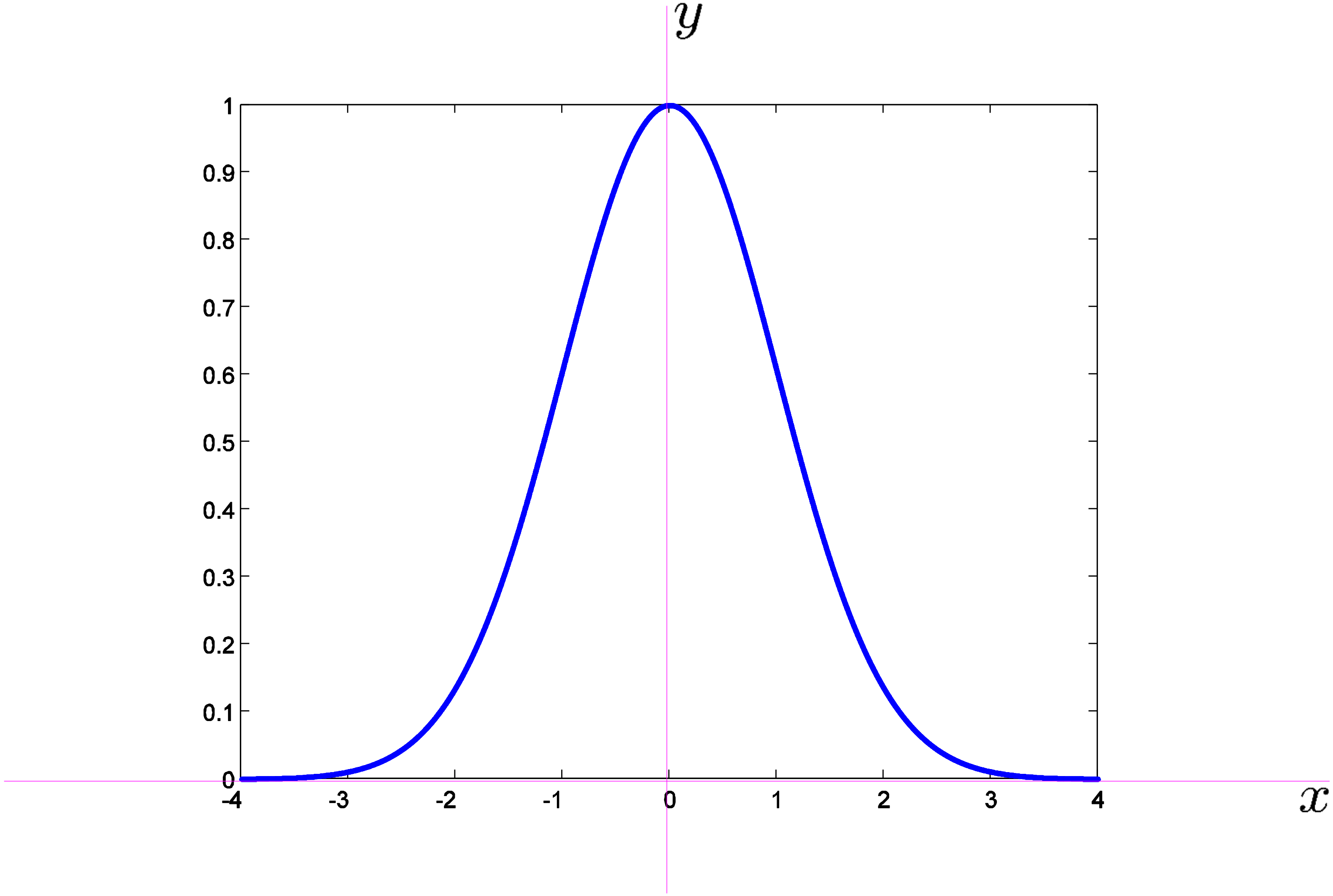

$$y = [\cos(x/\sqrt{10})]^{10}$$

$$\lim_{n\to\infty} [\cos(x/\sqrt{n})]^{n} = e^{-x^2/2}$$



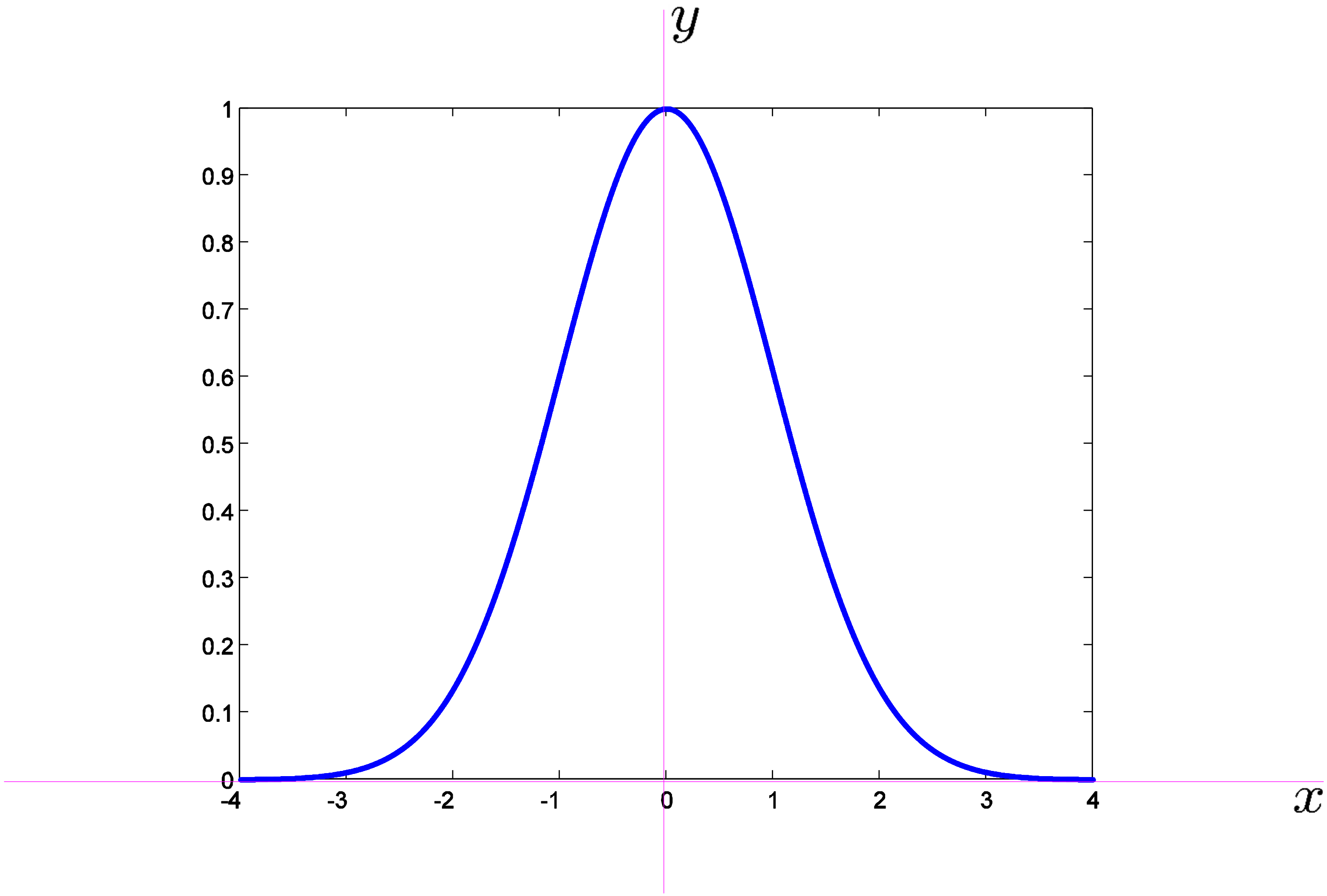

$$y = [\cos(x/\sqrt{1,000})]^{1,000}$$

$$\lim_{n\to\infty} [\cos(x/\sqrt{n})]^n = e^{-x^2/2}$$




$$y = [\cos(x/\sqrt{10,000})]^{10,000}$$

$$\lim_{n \to \infty} [\cos(x/\sqrt{n})]^n = e^{-x^2/2}$$




$$y = e^{-x^2/2}$$

$$\lim_{n\to\infty} [\cos(x/\sqrt{n})]^n = e^{-x^2/2}$$

STOP