# Financial Mathematics

## Topology

# Boundary of a subset of $\mathbb{R}^n$

**Definition:** Let $S$ be a subset of $\mathbb{R}^n$, and let $p \in \mathbb{R}^n$. We say that $p$ is a **boundary point** of $S$ if there are sequences $s_j$ in $S$ and $t_j$ in $\mathbb{R}^n \backslash S$ s.t. $s_j \to p$ and $t_j \to p$, as $j \to \infty$.

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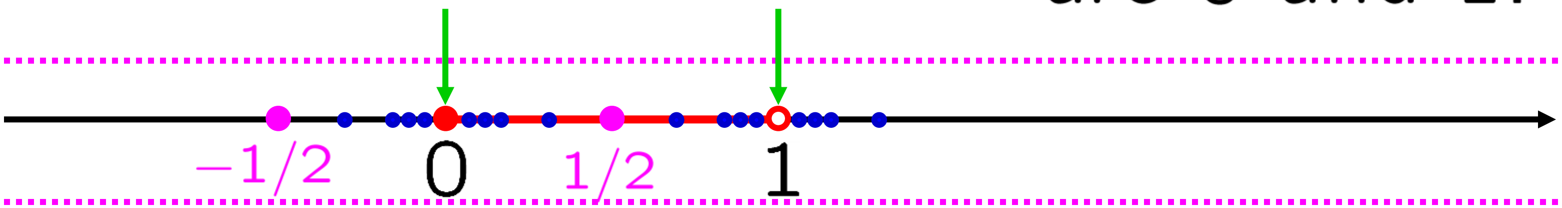
*e.g.*: Boundary points of $[0, 1) \subseteq \mathbb{R}$

are 0 and 1.




Question: Why not $-1/2$? no sequence in [0,1) approaches $-1/2$

Question: Why not $1/2$? no sequence outside [0,1) approaches $1/2$

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Question: What are the boundary points of $(3, 4] \times (1, 2) \subseteq \mathbb{R}^2$?

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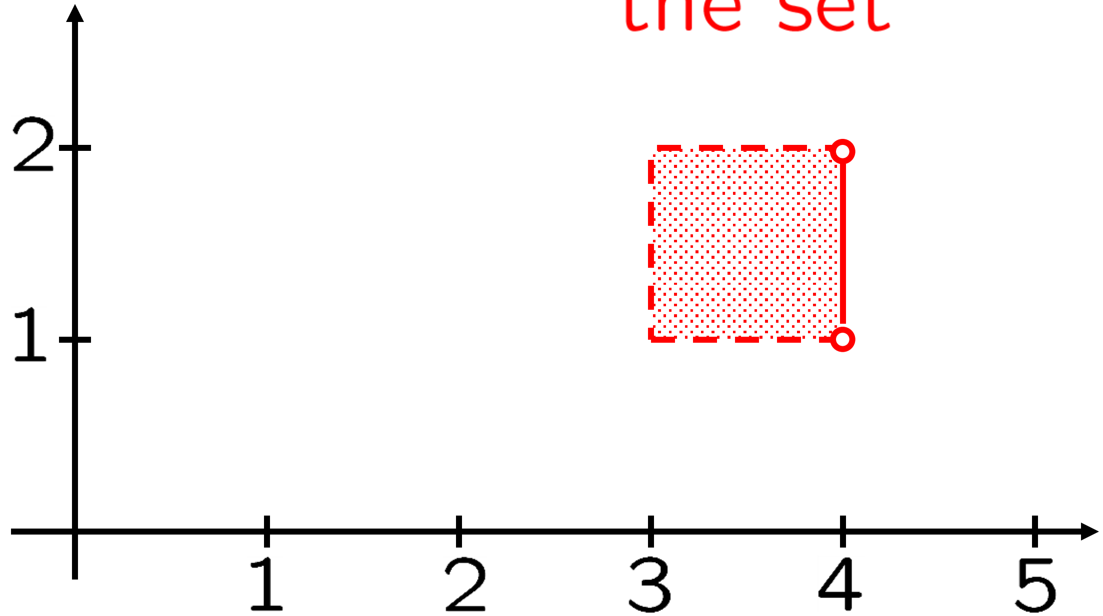
the set

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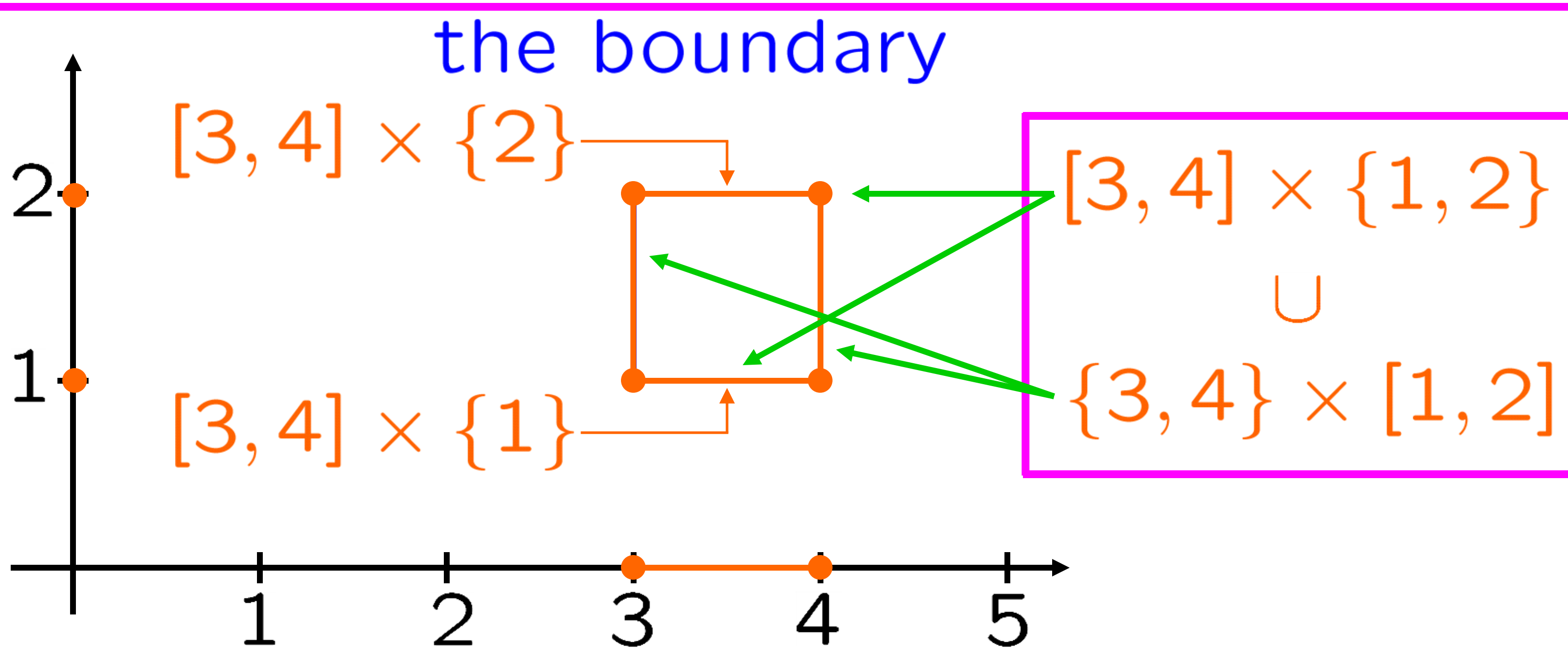


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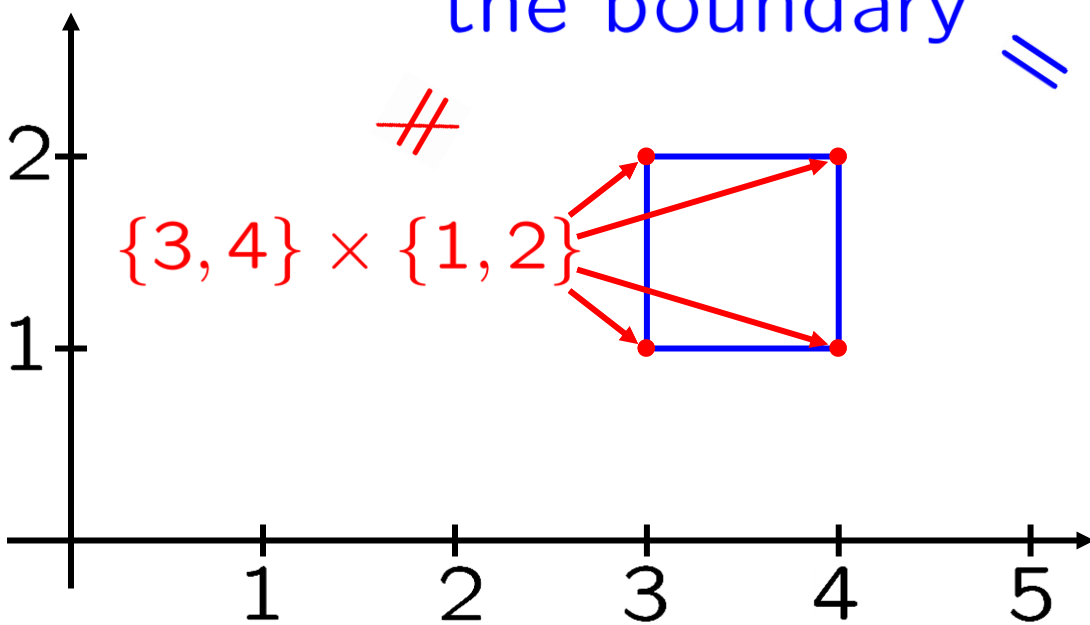


the boundary $=$ $[3,4] \times \{1,2\}$ $\cup$ $\{3,4\} \times [1,2]$

**Question:** What are the boundary points of $(3,4] \times (1,2) \subseteq \mathbb{R}^2$?

boundary of product $\neq$ product of boundaries

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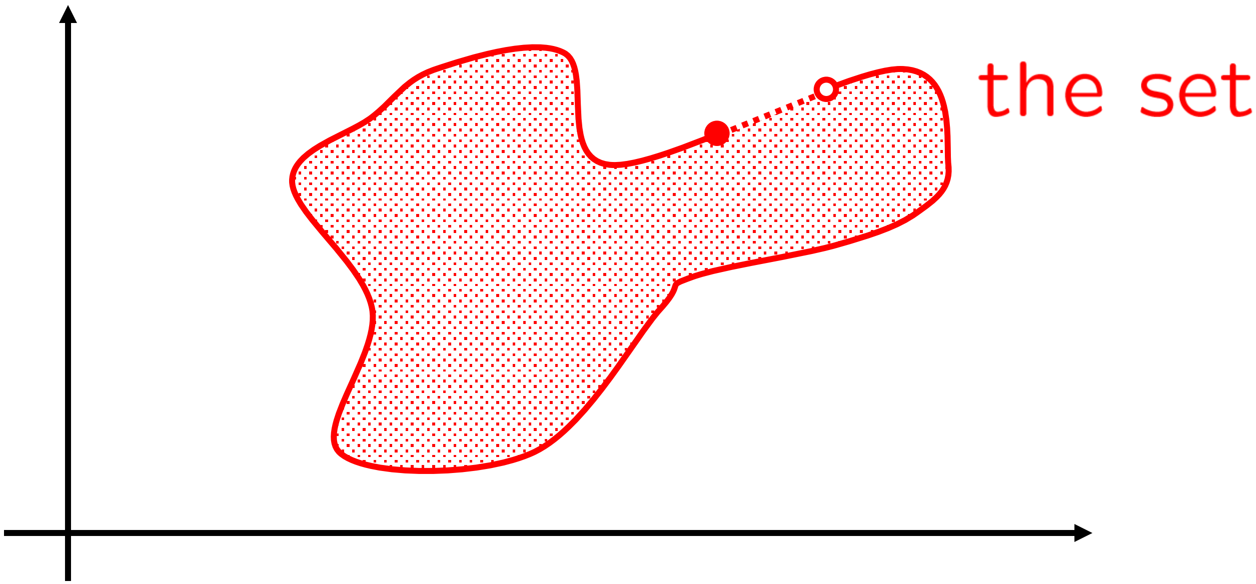


**Question:** What are the boundary points of the set shown above?

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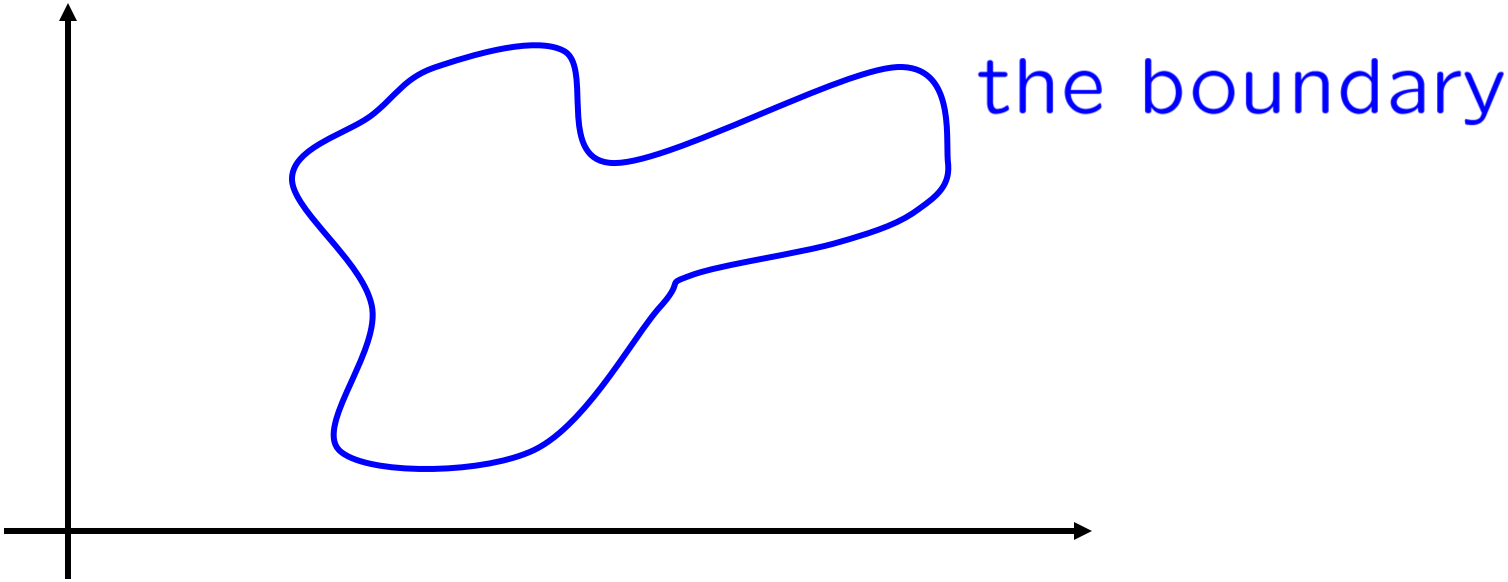
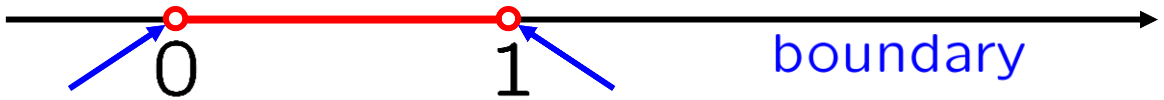



Question: What are the boundary points of the set shown above?

# Open, closed and compact sets

**Definition**: A subset $S$ of $\mathbb{R}^n$ is **open** if it contains none of its boundary points.

*e.g.*: $(0, 1)$

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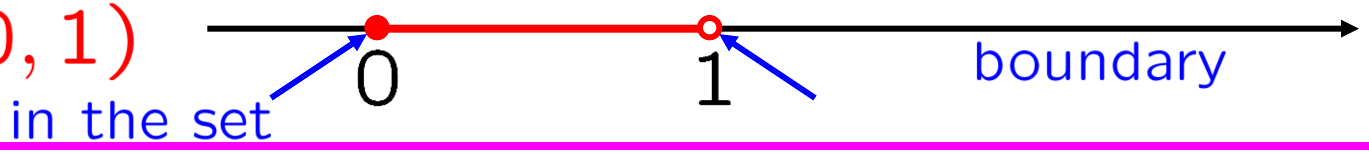


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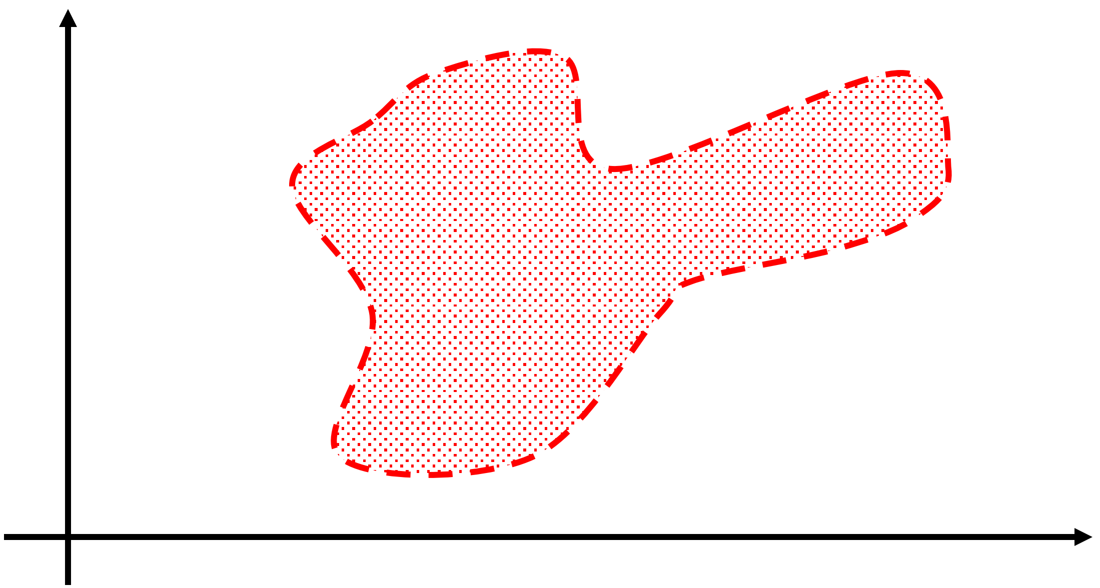

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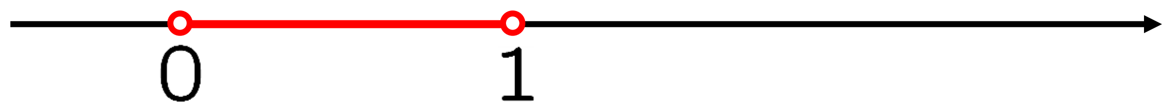


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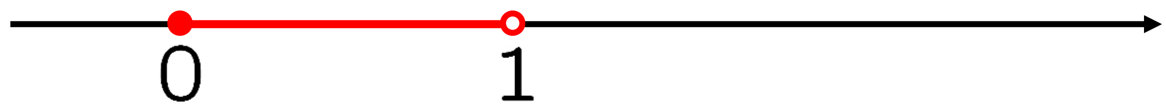


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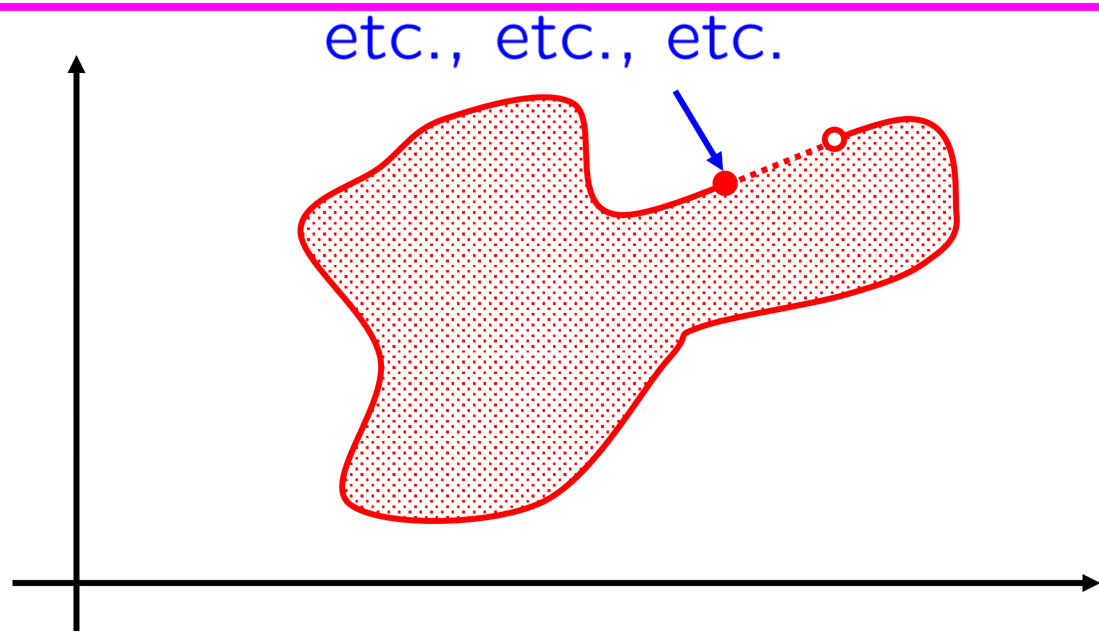

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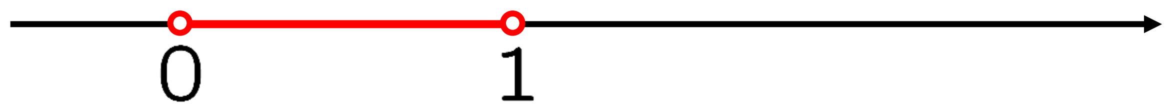


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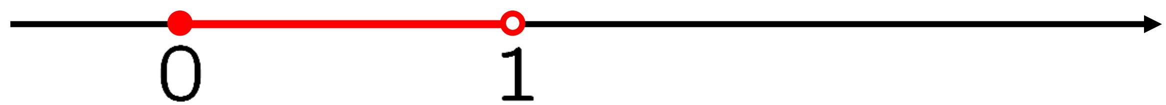


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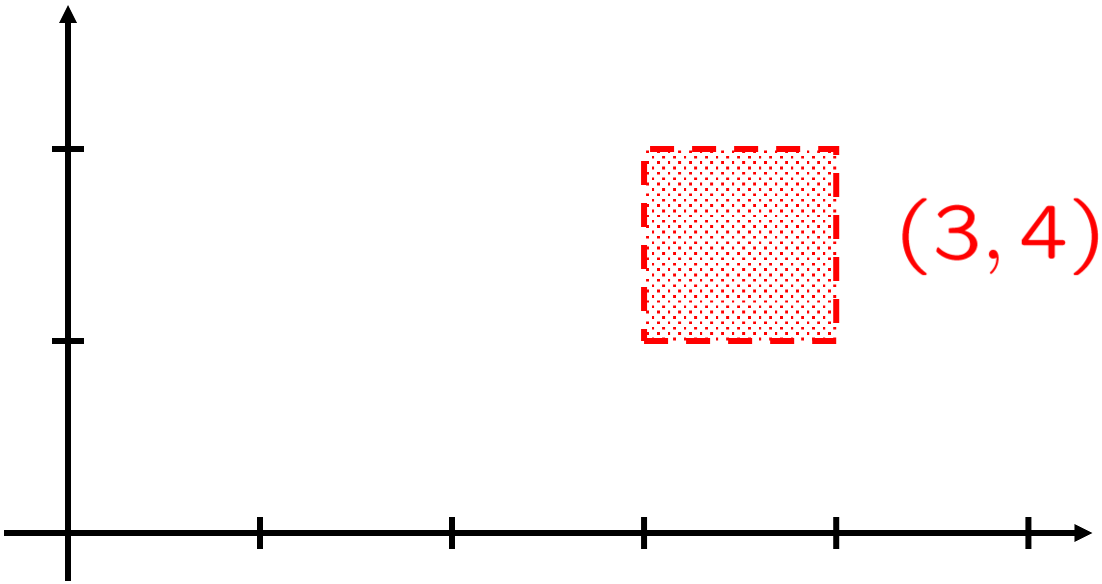

$(3,4) \times (1,2)$

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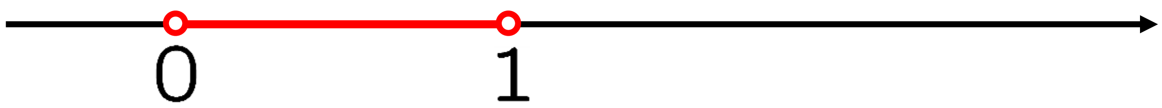


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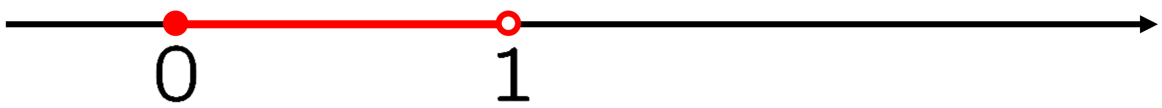


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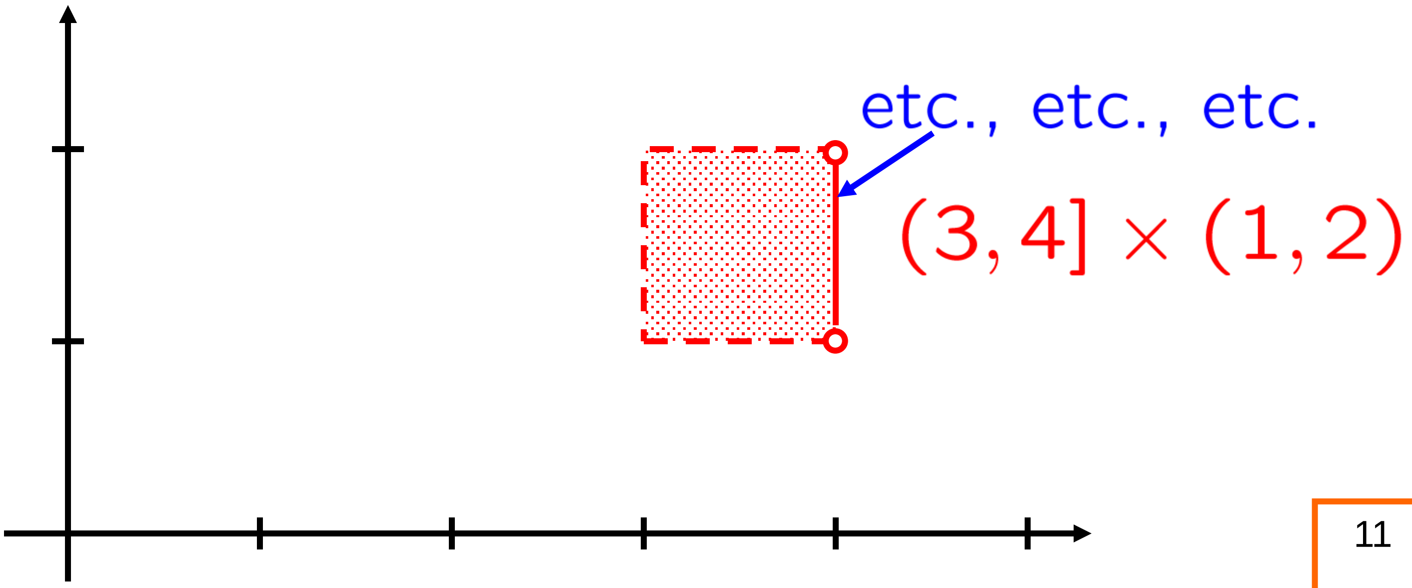

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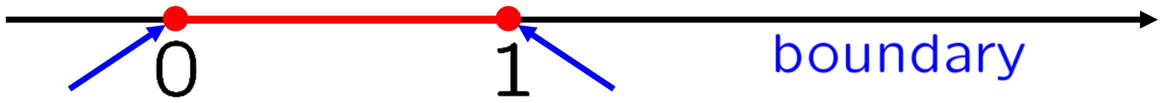


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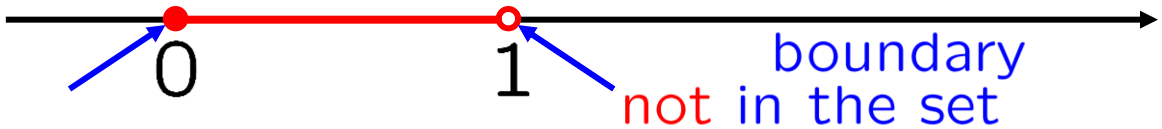


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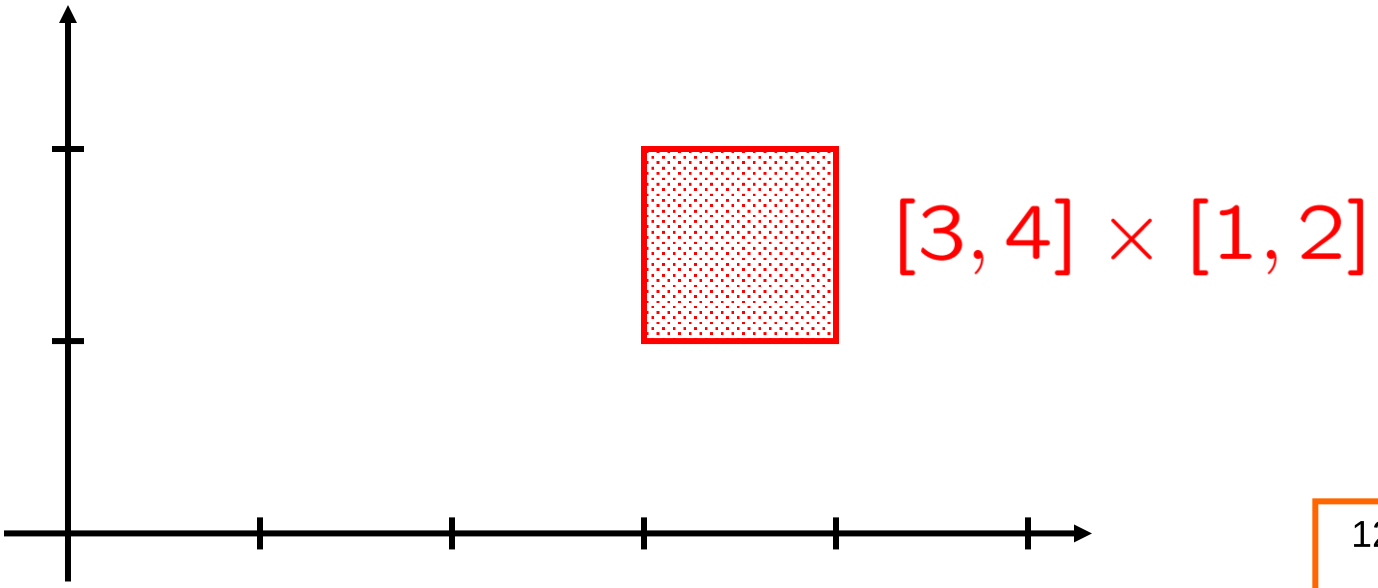

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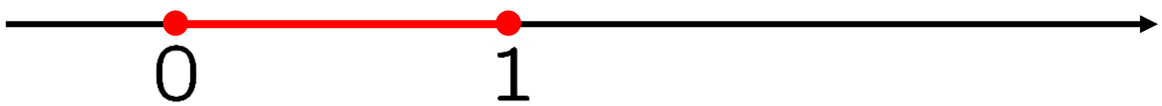


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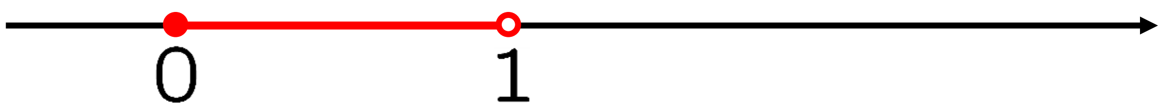


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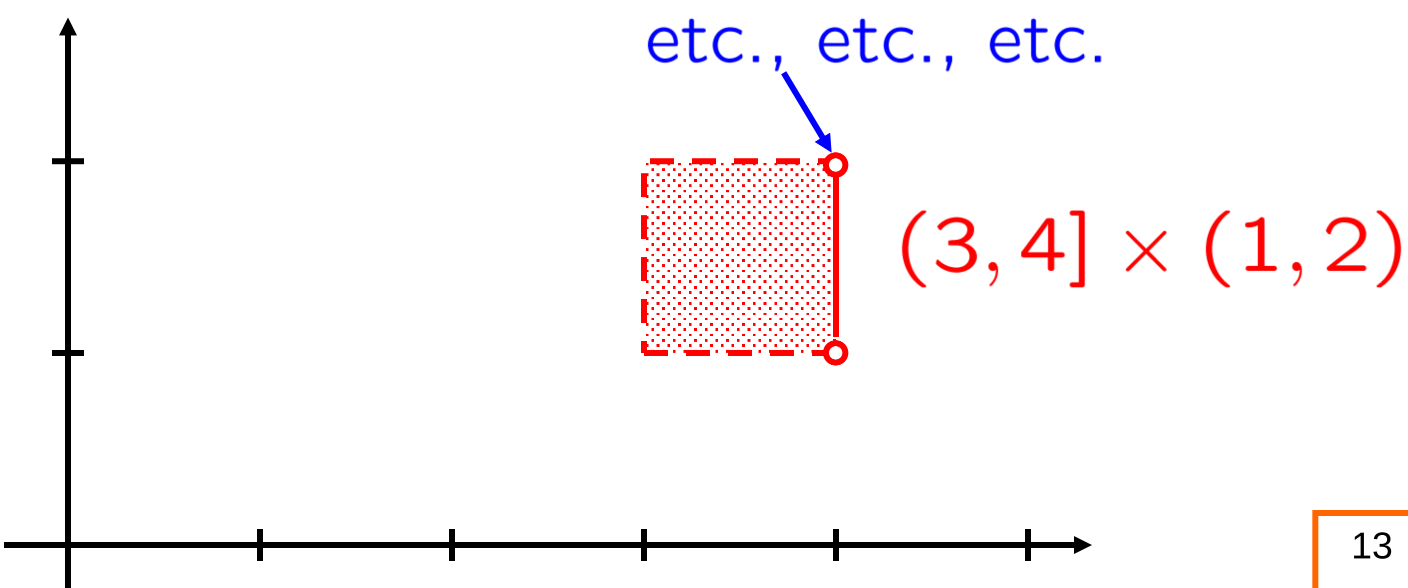

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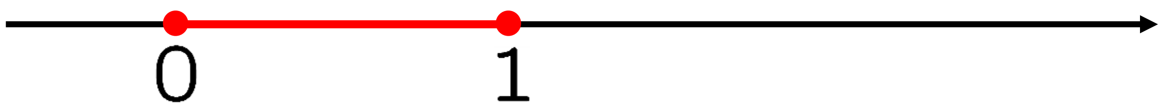


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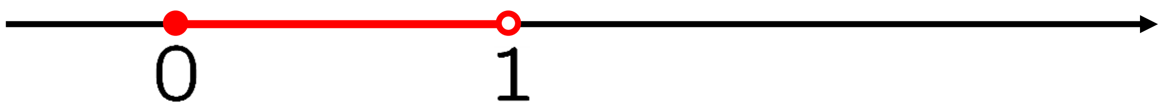


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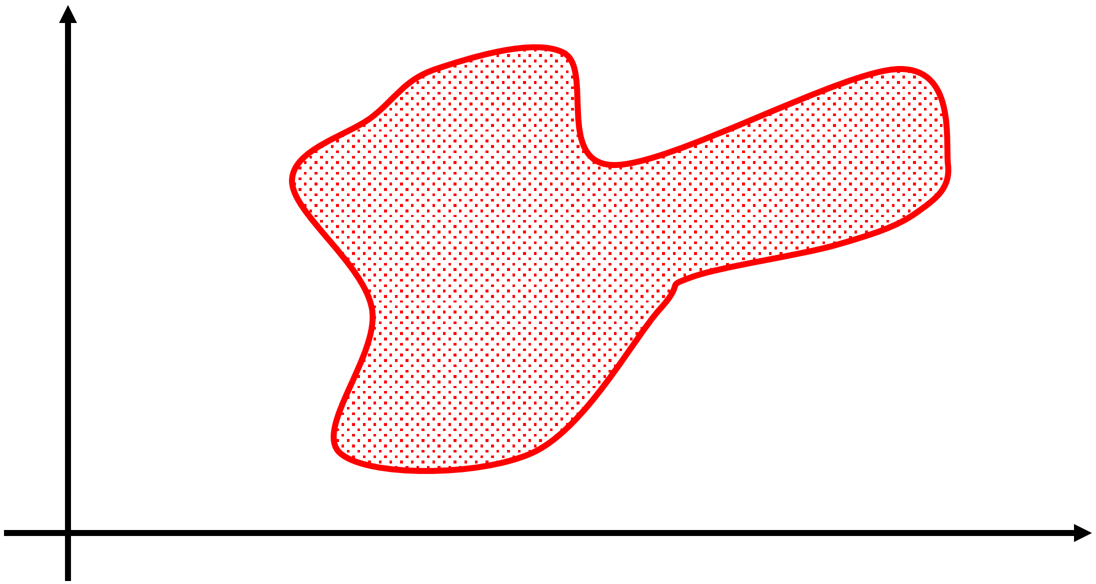

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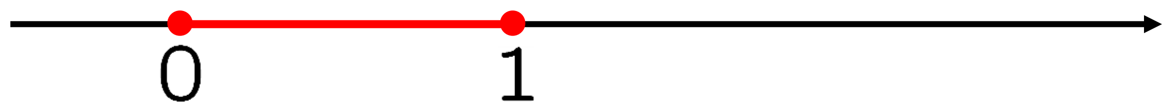


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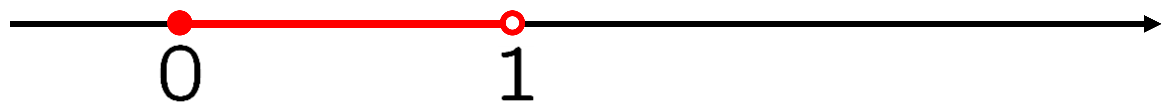
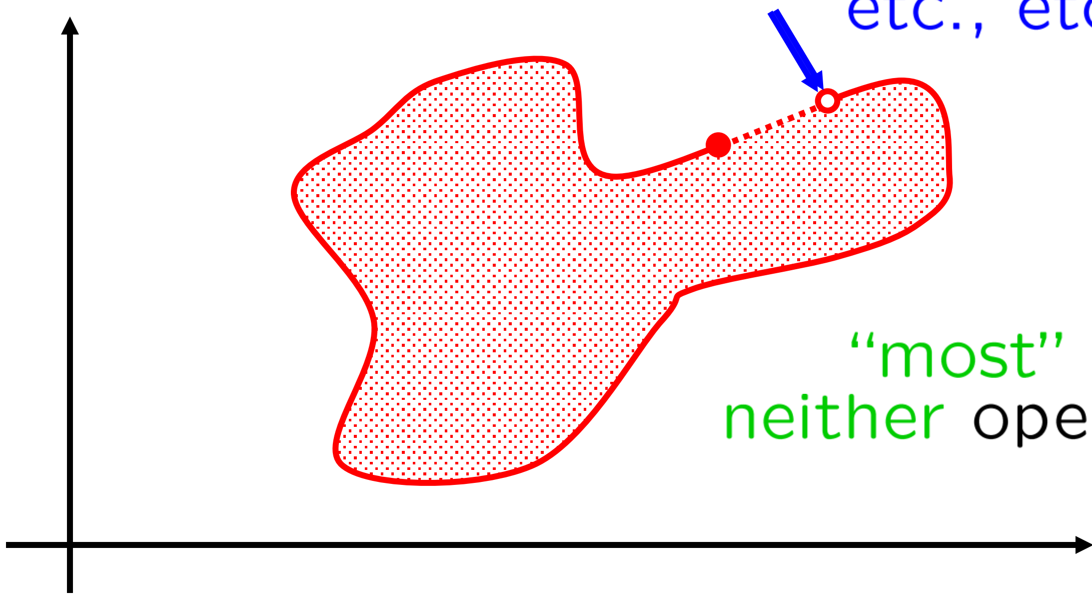



non-*e.g.*:

etc., etc., etc.

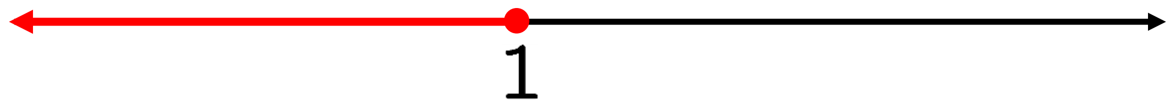


"most" sets are neither open nor closed

# Open, closed and compact sets

Definition: A subset $S$ of $\mathbb{R}^n$ is **closed** if it contains all of its boundary points.

*e.g.*: $(-\infty, 1]$

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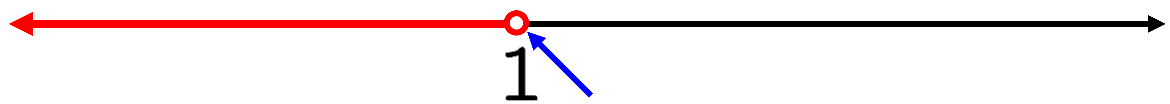


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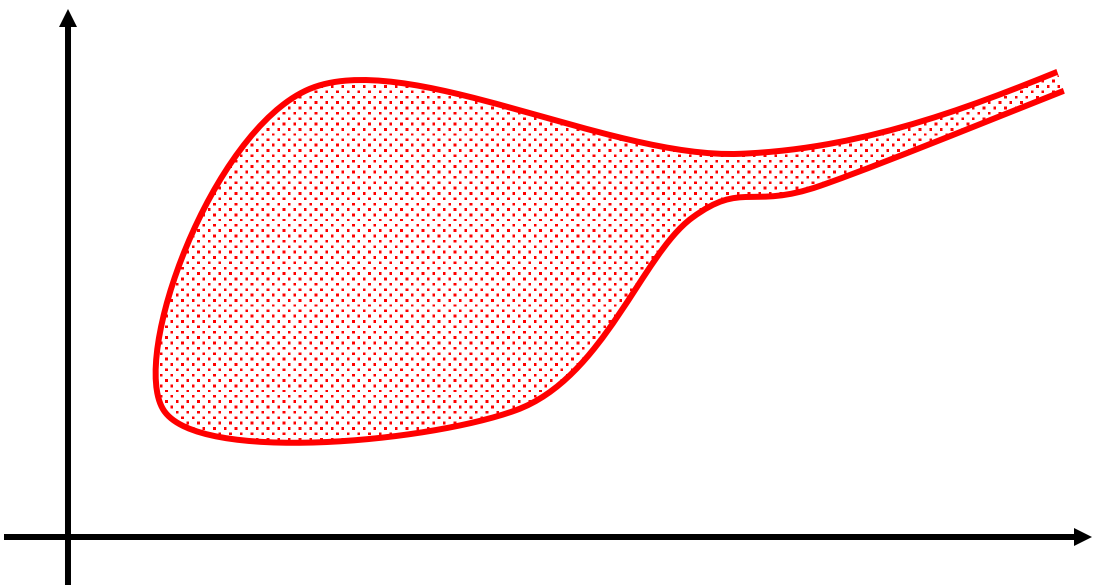

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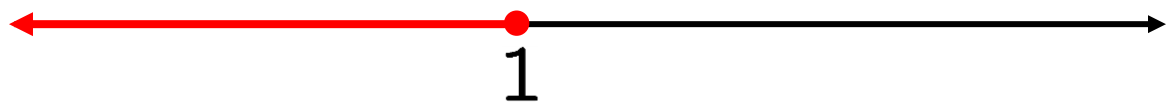


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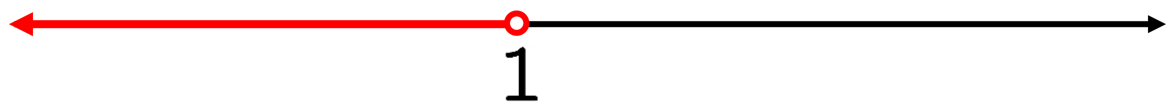


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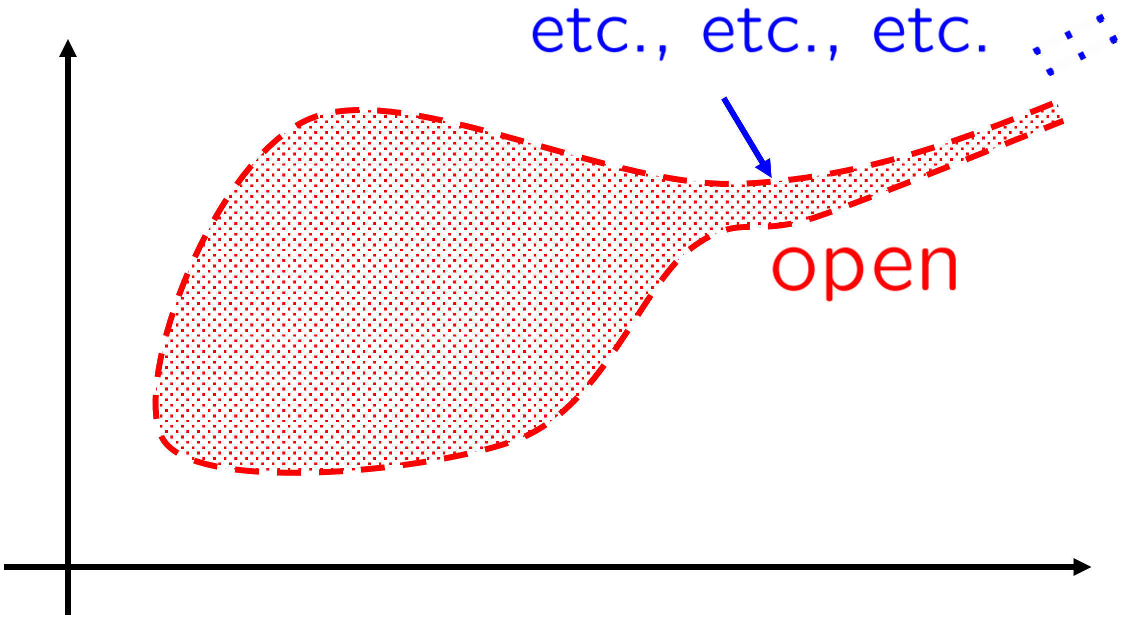


# Open, closed and compact sets

Definition: A subset $S$ of $\mathbb{R}^n$ is **closed** if it contains all of its boundary points.

Definition: A subset $S$ of $\mathbb{R}^n$ is **open** if it contains none of its boundary points.

Can a set be both closed and open?

clopen = closed and open

$\emptyset$ and $\mathbb{R}^n$ are both "clopen" subsets of $\mathbb{R}^n$.

There are no others.

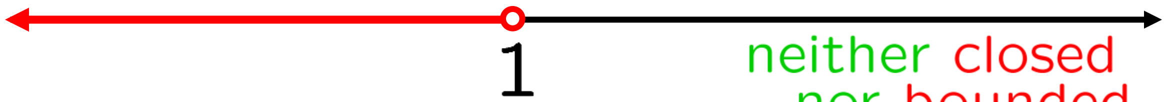
# Open, closed and compact sets

Definition: A subset $S$ of $\mathbb{R}^n$ is **compact** if it is both closed and bounded.

non-*e.g.*: $(-\infty, 1]$

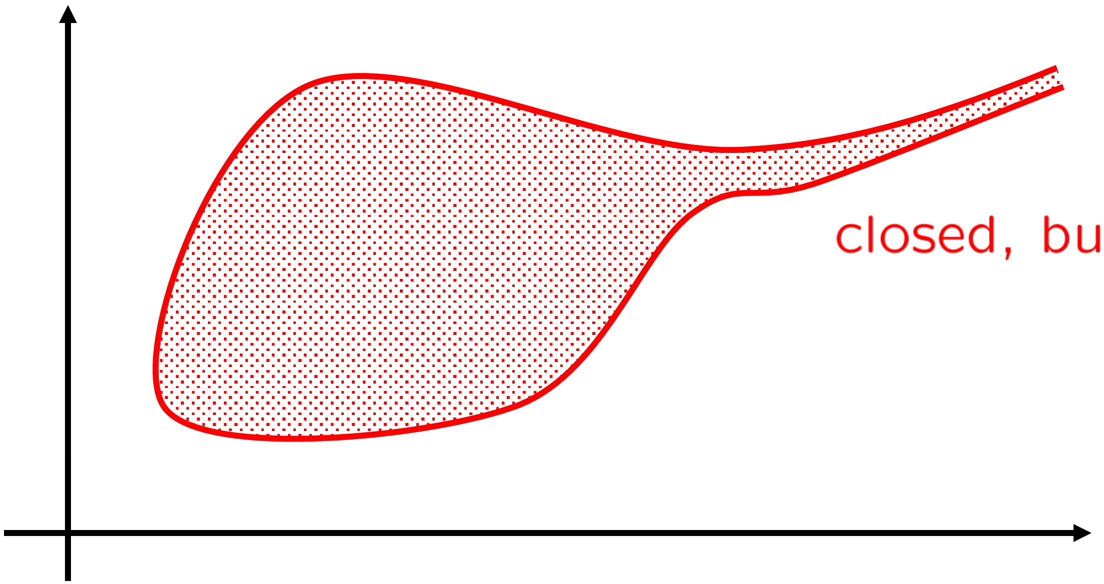

closed, but not bounded

non-*e.g.*: $(-\infty, 1)$




neither closed nor bounded

non-*e.g.*:



closed, but not bounded

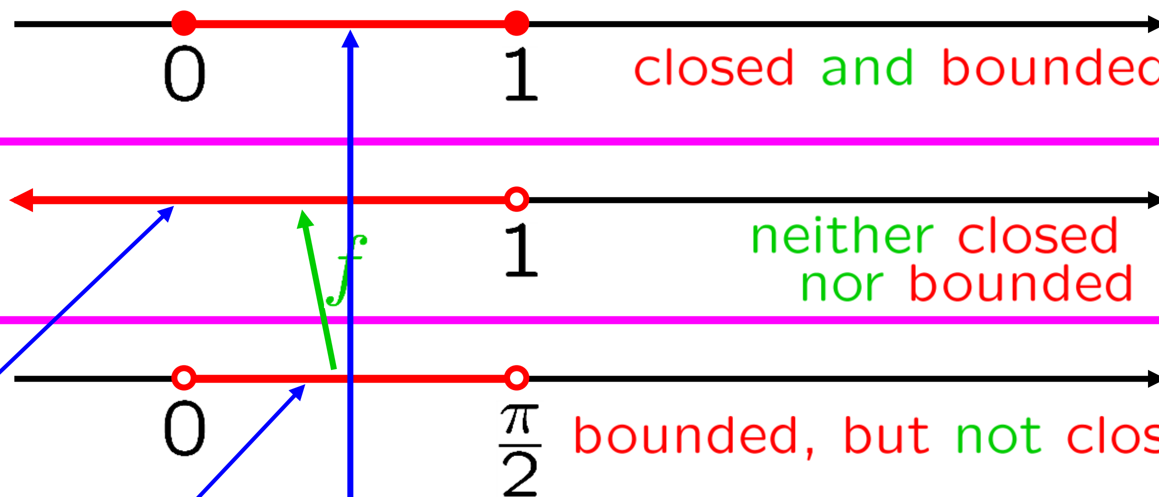
# Open, closed and compact sets

**Definition**: A subset $S$ of $\mathbb{R}^n$ is **compact** if it is *both* closed *and* bounded.

*e.g.*: $[0,1]$

non-*e.g.*: $(-\infty, 1)$

non-*e.g.*: $(0,1)$




homeomorphic
*i.e.*, the same,
to a topologist

$f(x) = 1 - [\tan x]$
contin., with contin. inverse,
*i.e.*, homeomorphism

**Intuition**: A compact set is not just bounded, but *SO* bounded, that all of its homeom. images continue to be bounded.

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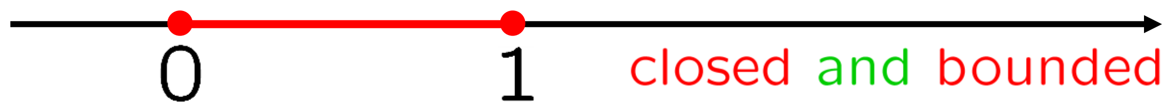
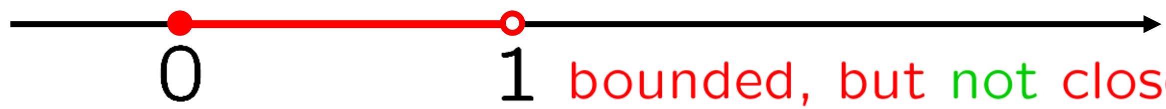



closed and bounded

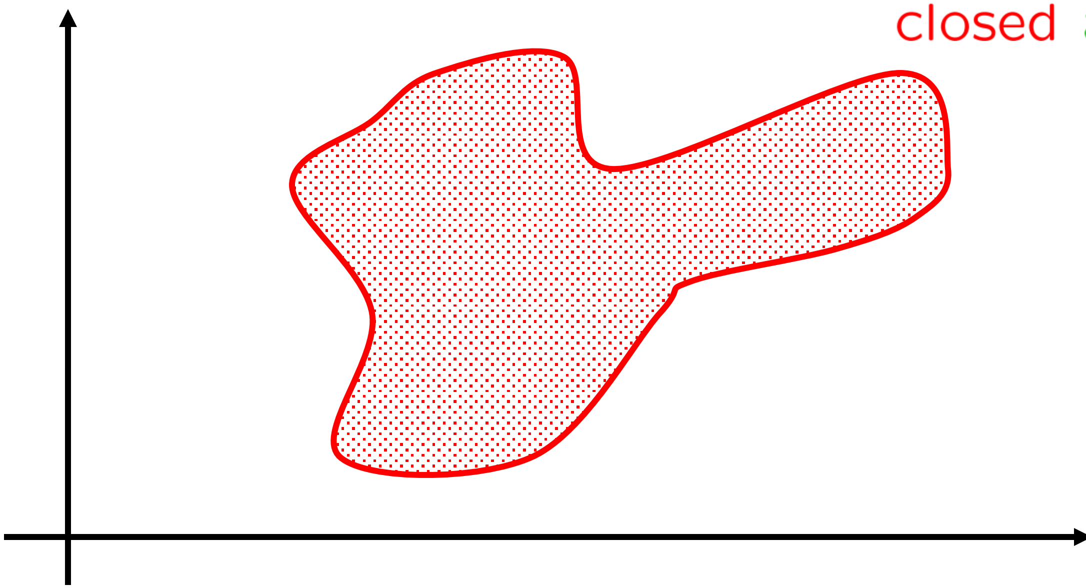
non-*e.g.*: $[0, 1)$




bounded, but not closed

*e.g.*:

closed and bounded



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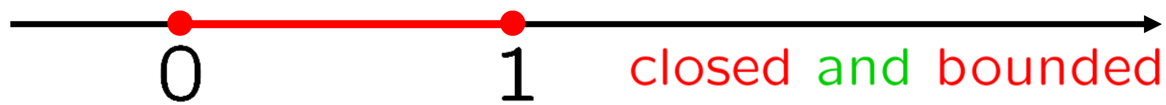
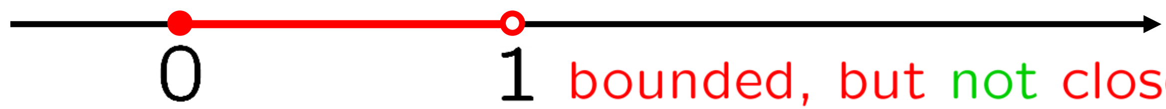



closed and bounded

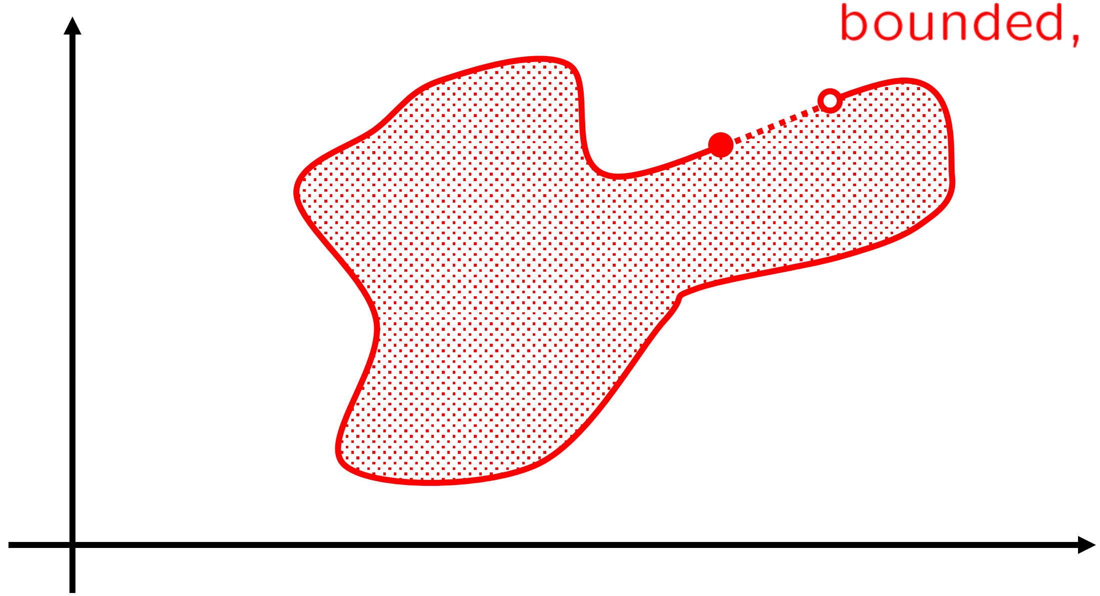
non-*e.g.*: $[0, 1)$




bounded, but not closed

non-*e.g.*:

bounded, but not closed



SKILL

Identify, from a picture, whether a given subset of $\mathbb{R}^n$ is open, closed, both or neither.

SKILL

Identify, from a picture, whether a given subset of $\mathbb{R}^n$ is compact.

For pictures, usually $n \leq 2$.

SKILL

Identify, from a description, whether a given subset of $\mathbb{R}^n$ is open, closed, both or neither.

*e.g.,* $[1,2] \times [3,4] \times [5,6]$

SKILL

Identify, from a description, whether a given subset of $\mathbb{R}^n$ is compact.

Question:

Are there any subsets of $\mathbb{R}$ that are both open and compact?

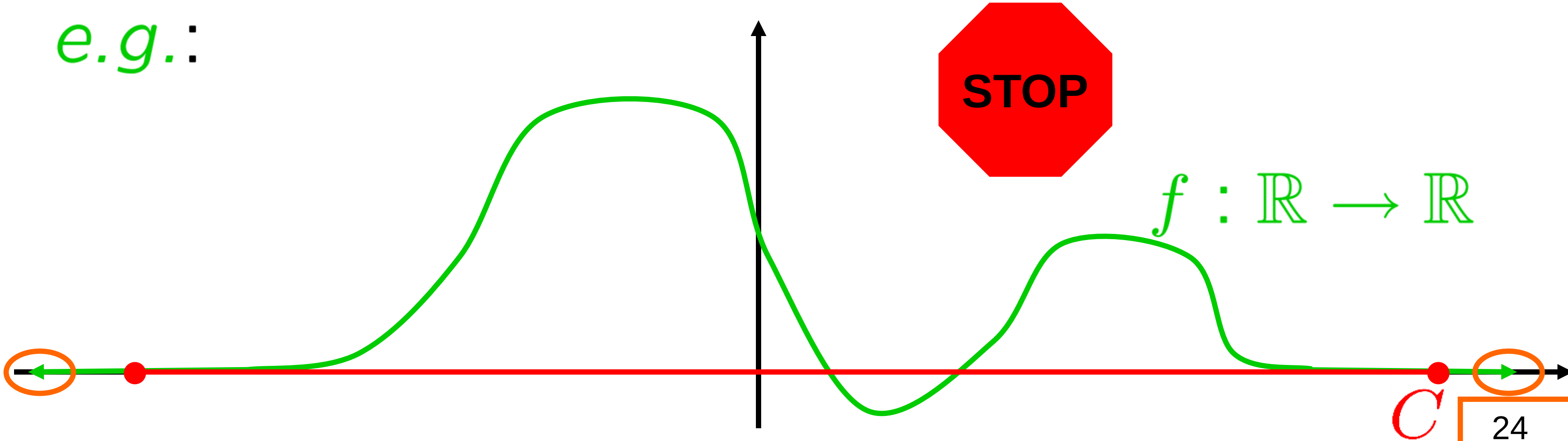
# Compactly supported functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

Def'n: Let $C \subseteq \mathbb{R}^n$ be a closed set.
We say that $f$ is **supported on** $C$ if,
for all $x \in \mathbb{R}^n \backslash C$, $\quad f(x) = 0$.

Def'n: We say $f$ is **compactly supported** or **has compact support** if there exists a compact set $C$ such that $f$ is supported on $C$.

*e.g.*:




$f$ is supported on $C$. $f$ has compact support.