Financial Mathematics

Basics of linear transformations

Linear Relationships

Suppose we are measuring six quantities

$$u,\ v,\ w,\ x,\ y,\ z.$$

We say that

y and z **depend linearly** on u, v, w and x

if there are constants

$$a,\ b,\ c,\ d,$$
$$e,\ f,\ g,\ h$$

such that, whenever we do a measurement, we find that

$$y = au + bv + cw + dx$$
$$z = eu + fv + gw + hx\,.$$

$$y = (a, b, c, d) \cdot (u, v, w, x)$$
$$z = (e, f, g, h) \cdot (u, v, w, x)$$

An exact linear relationship:

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$$

There are sometimes

exact linear relationships,

and sometimes, we recognize relationships that are "almost linear", with

$$
\begin{aligned}
y &\approx au + bv + cw + dx \\
z &\approx eu + fv + gw + hx\,,
\end{aligned}
$$

and the goal is to find $a, b, c, d,$ e, f, g, h

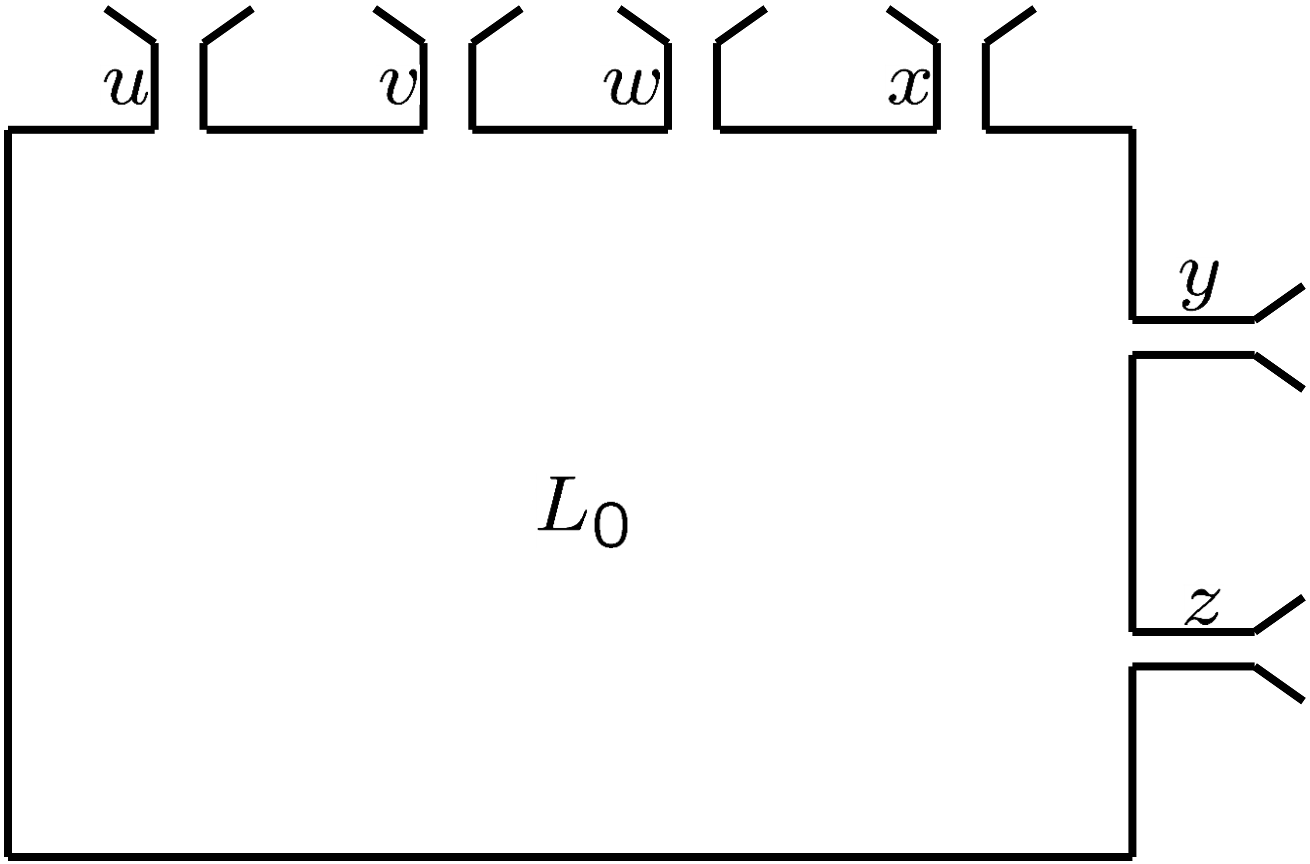
that make the approximation as close as possible.

In this lecture,

we'll study exact linear relationships.

E.g.,

$$y = 2u + 5v + 3w - 9x$$
$$z = 7u - 6v + 4w - 5x$$



$L_0 : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by

$$L_0(u, v, w, x) = (2u + 5v + 3w - 9x, \; 7u - 6v + 4w - 5x)$$

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Dimensions of M_0: 2×4

Key point: To any $k \times n$ matrix M, there is a function $L_M : \mathbb{R}^n \longrightarrow \mathbb{R}^k$.

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To get the ith entry in $L_{M_0}(u,v,w,x)$,
dot the ith row of M_0 with (u,v,w,x)

$M \in \mathbb{R}^{k\times n}$, $p \in \mathbb{R}^n$

To get the ith entry in $L_M(p)$,
dot the ith row of M with p

SKILL: Compute $L_M(p)$.

Definition: Let V be a subspace of $\mathbb{R}^n$.
Let W be a subspace of $\mathbb{R}^k$.
A function $F : V \to W$ is **linear**
if it respects the linear operations,
i.e., both of the following hold:

- for all $v, v' \in V$,
$$F(v + v') = F(v) + F(v'),$$
- for all scalars c, for all $v \in V$,
$$F(cv) = c[F(v)],$$

i.e., F respects linear combinations,
i.e., for all integers $k > 0$,
for all scalars $c_1, \ldots, c_k$,
for all vectors $v_1, \ldots, v_k \in V$,
$$F(c_1 v_1 + \cdots + c_k v_k) = c_1[F(v_1)] + \cdots + c_k[F(v_k)].$$

E.g.: I'm thinking of a secret 3×4 matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

Definition: L_M is called the

linear function corresponding to M.

Game: I pick and tell you integers $k, n > 0$.

I pick a secret matrix $M \in \mathbb{R}^{k \times n}$.

Your goal is to find M.

You pick and tell me finite sequence

$$v_1, \ldots, v_p \in \mathbb{R}^n$$

and I tell you $L_M(v_1), \ldots, L_M(v_p)$.

How can you figure out M?

E.g.: I'm thinking of a secret 3×4 matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

Then $L_M(w, x, y, z)$ is equal to

$$\begin{aligned} (aw + bx + cy + dz, \\ ew + fx + gy + hz, \\ iw + ix + ky + lz) \end{aligned}$$

suggestions??

Then $L_M(1, 0, 0, 0) = (a, e, i)$, so you can find the first column of M. The other three columns can be found by asking for

$L_M(0, 1, 0, 0)$, $L_M(0, 0, 1, 0)$, $L_M(0, 0, 0, 1)$.

Key point to remember:

The entries in the jth column of M are the same as the entries in

$$L_M(0, \ldots, 0, 1, 0, \ldots, 0).$$

jth entry

Notation:

The matrix of a linear transformation

$$A : \mathbb{R}^n \to \mathbb{R}^k$$

is denoted $\boxed{[A]}$.

Note: $L_{[A]} = A$ and $[L_M] = M$

Fact: A function $F : \mathbb{R}^n \to \mathbb{R}^k$ is linear iff there exists $M \in \mathbb{R}^{k \times n}$ such that $F = L_M$.

Fact: $L_M = L_{M'}$ implies $M = M'$.

Definition:

Let V, W be subspaces of $\mathbb{R}^n$, $\mathbb{R}^k$, resp.

A function $F : V \to W$ is an **isomorphism** (or a **vector space isomorphism**) if it's a linear bijection.

Key idea:

For any fact you've verified about V, there's a corresponding fact about W, and vice versa.

e.g.: If you know that every basis of V has three vectors, then the same must be true of W.

The kernel and image of a linear function

Let V, W be subspaces of $\mathbb{R}^n$, $\mathbb{R}^k$, resp.
Let $L : V \to W$ be a linear function.

The **kernel of** L is

$$\boxed{\text{ker}(L)} := \{v \in V \,|\, L(v) = 0\}.$$

Fact:
$L : V \to W$ is one-to-one iff $\text{ker}(L) = \{0\}$.

The **image of** L is

$$\boxed{L(V)} := \{L(v) \in W \,|\, v \in V\}.$$

Observation:
$L : V \to W$ is onto iff $L(V) = W$.

Fact: Kernels and images are subspaces.

The **kernel of** L is

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Pf: $(\Rightarrow)$

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Pf: $(\Rightarrow)$ Say L is 1-1. Want: $\ker(L) = \{0\}$.

∀linear L, $L(0) = 0$

Say $v \in \ker(L)$. Want: $v = 0$.

$$L(v) = 0 = L(0)$$

$$v = 0$$

☺

$(\Leftarrow)$ Say $\ker(L) = \{0\}$. Want: L is 1-1.

Say $L(v) = L(v')$. Want: $v = v'$.

∀linear L, $\forall v, v'$, $L(v - v') = (L(v)) - (L(v'))$

$$L(v - v') = (L(v)) - (L(v')) = 0$$

$$v - v' \in \ker(L) = \{0\}$$

$$v - v' = 0$$

$$v = v'$$

☺ QED

m-dimensional Euclidean space

Definition:

Let V be a subspace of $\boxed{\mathbb{R}^m}$.

An **ordered basis of** V

is, for some integer $d \geq 1$,

an ordered d-tuple $(v_1, \ldots, v_d) \in V^d$

such that $\{v_1, \ldots, v_d\}$ is a basis of V.

Note: $(v_1, \ldots, v_d) \in V^d \subseteq (\mathbb{R}^m)^d$

---

e.g.:

$S := \langle (1,3,4,2), \quad (2,1,2,-1), \quad (4,7,10,3) \rangle$

$\{(1,3,4,2), \quad (2,1,2,-1)\}$ is a basis of S.

$((1,3,4,2), \quad (2,1,2,-1))$ is an ordered basis of S.

$((2,1,2,-1), \quad (1,3,4,2))$ is a different one.

Ordered bases $\leftrightarrow$ isomorphisms w/ Eucl. space

Fact:

Let $(v_1, \ldots, v_d) \in V^d$ be an ordered basis of a subspace V of some Euclidean space.
Then the function $F : \mathbb{R}^d \to V$ defined by

$$F(a_1, \ldots, a_d) = a_1 v_1 + \cdots + a_d v_d$$

is a vector space isomorphism.

Proof: Linearity is an exercise.

Onto because $\{v_1, \ldots, v_d\}$ is a spanning set.

Want: F is one-to-one.

Want: $\ker(F) = \{0\}$.

$(v_1, \ldots, v_d) \in V^d$ is an ordered basis of V.

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$$a_1 v_1 + \cdots + a_d v_d = 0$$

$\{v_1, \ldots, v_d\}$ l.i

NO NONTRIVIAL l.c. OF $v_1, \ldots, v_d$ IS EQUAL TO 0.

$$a_1 = \cdots = a_d = 0$$

$$(a_1, \ldots, a_d) = 0 \in \mathbb{R}^d.$$

QED

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Then S is isomorphic to $\mathbb{R}^2$,
so anything we know about $\mathbb{R}^2$
translates into knowledge about S.

More on this later . . .

STOP