Financial Mathematics

Introduction to row and column operations

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & & 10\,u & = & 5 \\ 4s & - & 12t & + & 12\,u & = & 16 \\ 3s & - & 7t & + & 4\,u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\begin{array}{rcrcrcr} 4s & - & 12t & + & 12\,u & = & 16 \\ & & 5t & - & 10\,u & = & 5 \\ 3s & - & 7t & + & 4\,u & = & 9 \end{array}$$

divide 1st by 4

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3\,u & = & 4 \\ & & 5t & - & 10\,u & = & 5 \\ 3s & - & 7t & + & 4\,u & = & 9 \end{array}$$

add $-3\times$(1st) to 3rd

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3\,u & = & 4 \\ & & 5t & - & 10\,u & = & 5 \\ & & 2t & - & 5\,u & = & -3 \end{array}$$

divide 2nd by 5

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3\,u & = & 4 \\ & & t & - & 2\,u & = & 1 \\ & & 2t & - & 5\,u & = & -3 \end{array}$$

add $-2\times$(2nd) to 3rd

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\vdots$$

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3u & = & 4 \\ & & t & - & 2u & = & 1 \\ & & 2t & - & 5u & = & -3 \end{array}$$

add $-2\times$(2nd) to 3rd

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3u & = & 4 \\ & & t & - & 2u & = & 1 \\ & & & - & u & = & -5 \end{array}$$

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3u & = & 4 \\ & & t & - & 2u & = & 1 \\ & & 2t & - & 5u & = & -3 \end{array}$$

add $-2\times$(2nd) to 3rd

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\vdots$$

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3u & = & 4 \\ & & t & - & 2u & = & 1 \\ & & 2t & - & 5u & = & -3 \end{array}$$

add $-2\times$(2nd) to 3rd

$$\begin{array}{rcrcrcr} s & - & 3t & + & 3u & = & 4 \\ & & t & - & 2u & = & 1 \\ & & & - & u & = & -5 \end{array}$$

to 1st, add $3\times$(2nd)

$$\begin{array}{rcrcrcr} s & & & - & 3u & = & 7 \\ & & t & - & 2u & = & 1 \\ & & & - & u & = & -5 \end{array}$$

multiply 3rd by -1

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\vdots$$

$$\begin{array}{rcrcrcr} s & & & - & 3u & = & 7 \\ & & t & - & 2u & = & 1 \\ & & & - & u & = & -5 \end{array}$$

multiply 3rd by -1

$$\begin{array}{rcrcrcr} s & & & - & 3u & = & 7 \\ & & t & - & 2u & = & 1 \\ & & & & u & = & 5 \end{array}$$

$$\begin{array}{rcrcrcr} s & & & - & 3u & = & 7 \\ & & t & - & 2u & = & 1 \\ & & & - & u & = & -5 \end{array}$$

multiply 3rd by -1

Solving a linear system

$$\begin{aligned} 5t - 10u &= 5 \\ 4s - 12t + 12u &= 16 \\ 3s - 7t + 4u &= 9 \end{aligned}$$

interchange 1st & 2nd

$$\vdots$$

$$\begin{aligned} s - 3u &= 7 \\ t - 2u &= 1 \\ -u &= -5 \end{aligned}$$

multiply 3rd by -1

$$\begin{aligned} s - 3u &= 7 \\ t - 2u &= 1 \\ u &= 5 \end{aligned}$$

to 1st,
to 2nd,
add 3,2× (3rd)

$$\begin{aligned} s &= 22 \\ t &= 11 \\ u &= 5 \end{aligned}$$

DONE!

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\begin{bmatrix} 0 & 5 & -10 & 5 \\ 4 & -12 & 12 & 16 \\ 3 & -7 & 4 & 9 \end{bmatrix}$$

interchange 1st & 2nd

$$\downarrow$$

$$\begin{bmatrix} 4 & -12 & 12 & 16 \\ 0 & 5 & -10 & 5 \\ 3 & -7 & 4 & 9 \end{bmatrix}$$

divide 1st by 4

$$\downarrow$$

$$\vdots$$

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\begin{bmatrix} 0 & 5 & -10 & 5 \\ 4 & -12 & 12 & 16 \\ 3 & -7 & 4 & 9 \end{bmatrix}$$

interchange 1st & 2nd

$$\downarrow \ \vdots \ \downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 22 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10\,u & = & 5 \\ 4s & - & 12t & + & 12\,u & = & 16 \\ 3s & - & 7t & + & 4\,u & = & 9 \end{array}$$

interchange
1st & 2nd

$$\begin{bmatrix} 0 & 5 & -10 & 5 \\ 4 & -12 & 12 & 16 \\ 3 & -7 & 4 & 9 \end{bmatrix}$$

interchange
1st & 2nd

$$\downarrow \vdots \downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 22 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

DONE!

Solving a linear system

$$\begin{array}{rcrcrcr} & & 5t & - & 10u & = & 5 \\ 4s & - & 12t & + & 12u & = & 16 \\ 3s & - & 7t & + & 4u & = & 9 \end{array}$$

interchange 1st & 2nd

$$\begin{bmatrix} 1 & 0 & 0 & 22 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

DONE!

$$\begin{array}{rcrcrcr} s & & & & & = & 22 \\ & & t & & & = & 11 \\ & & & & u & = & 5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 22 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

DONE!

Primary elementary row operations

1. Multiply a row by a nonzero constant
2. Add one row to another

e.g.:

Add second row to third

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix}$$

Multiply first row by -3

$$\rightarrow \begin{bmatrix} -6 & -15 & -9 & 27 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix}$$

KEY POINT:

Elem. row operations are left multiplications

e.g.:

Multiply first row by -3

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -6 & -15 & -9 & 27 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elem. matrix

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -15 & -9 & 27 \\ 7 & -6 & 4 & -5 \\ 10 & -8 & 4 & -1 \end{bmatrix}$$

Building secondary row operations

3. Choose a row.
   Multiply chosen row by a constant,
   and add the result to some other row,
   without changing the chosen row.

The two rows cannot be the same.

$$-1/5\times R2\rightarrow R1$$

$$A = \begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 & -2 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix} = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

$$EA = B$$

Building secondary row operations

3. Choose a row.
   Multiply chosen row by a constant,
   and add the result to some other row,
   without changing the chosen row.

The two rows cannot be the same.

A SECONDARY ROW OPERATION CAN BE OBTAINED FROM PRIMARY ROW OPERATIONS.

Multiply chosen row by a constant,
add chosen row (after const. mult.)
to the other row,
divide chosen row by the constant.

(If const.$=0$, don't do anything.)

Primary elementary row operations

1. Multiply a row by a nonzero constant
2. Add one row to another

Building secondary row operations

4. Interchange two rows.

R2↔R1

$$A = \begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 20 & 15 & -5 \\ 0 & 5 & 1 & -3 \\ 7 & -2 & 0 & 6 \end{bmatrix} = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

$$EA = B$$

Building secondary row operations

4. Interchange two rows.

A SECONDARY ROW OPERATION CAN BE OBTAINED FROM PRIMARY ROW OPERATIONS.

Call them "first row" and "second row".
Add first row to second,
multiply new second row by -1,
add new new second row to first,
multiply new first row by -1,
add new new first row
to new new second,
multiply new new new second row by -1.

Primary elementary row operations

1. Multiply a row by a nonzero constant
2. Add one row to another

Add first row to second,

$$\begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & 5 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

Add first row to second,

Add first row to second,
multiply new second row by -1,
add new new second row to first,
multiply new first row by -1, . . .

$$\begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 25 & 16 & -8 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 5 & -20 & -15 & 5 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 5 & 1 & -3 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -5 & 20 & 15 & -5 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

. . . add new new first row to new new second, multiply new new new second row by -1.

$$\begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 25 & 16 & -8 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 5 & -20 & -15 & 5 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 5 & 1 & -3 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -5 & 20 & 15 & -5 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 20 & 15 & -5 \\ 0 & -5 & -1 & 3 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

. . . add new new first row to new new second, multiply new new new second row by -1.

$$\begin{bmatrix} 0 & 5 & 1 & -3 \\ -5 & 20 & 15 & -5 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\downarrow \quad \vdots \quad \downarrow$$

$$\begin{bmatrix} -5 & 20 & 15 & -5 \\ 5 & -25 & -16 & 8 \\ 7 & -2 & 0 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} -5 & 20 & 15 & -5 \\ 0 & -5 & -1 & 3 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

$$\uparrow$$

$$\begin{bmatrix} -5 & 20 & 15 & -5 \\ 0 & 5 & 1 & -3 \\ 7 & -2 & 0 & 6 \end{bmatrix}$$

Primary elementary column operations

1. Multiply a column by a nonzero constant
2. Add one column to another

e.g.:

Add second column to third

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 8 & -9 \\ 7 & -6 & -2 & -5 \\ 3 & -2 & -2 & 4 \end{bmatrix}$$

Multiply first column by -3

$$\begin{bmatrix} -6 & 5 & 8 & -9 \\ -21 & -6 & -2 & -5 \\ -9 & -2 & -2 & 4 \end{bmatrix}$$

Building secondary column operations

3. Choose a column.
   Multiply chosen column by a constant,
   and add the result to some other column,
   without changing the chosen column.
   The two columns cannot be the same.

4. Interchange two columns.

Elem. col. operations are right multiplications

e.g.: **Add second column to third**

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 8 & -9 \\ 7 & -6 & -2 & -5 \\ 3 & -2 & -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elem. matrix

$$\begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \\ 3 & -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 & -9 \\ 7 & -6 & -2 & -5 \\ 3 & -2 & -2 & 4 \end{bmatrix}$$

Elem. col. operations are right multiplications

e.g.:

Multiply first column by -3

$$\begin{bmatrix} 2 & 5 & 8 & -9 \\ 7 & -6 & -2 & -5 \\ 3 & -2 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} -6 & 5 & 8 & -9 \\ -21 & -6 & -2 & -5 \\ -9 & -2 & -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elem. matrix

$$\begin{bmatrix} 2 & 5 & 8 & -9 \\ 7 & -6 & -2 & -5 \\ 3 & -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 5 & 8 & -9 \\ -21 & -6 & -2 & -5 \\ -9 & -2 & -2 & 4 \end{bmatrix}$$

Elem. matrices are invertible

Multiply first row by -3

e.g.: $I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: A$

Elem. matrix

Multiply first row by $-1/3$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: B$

Exercise: Check that $AB = BA = I$.

Elem. matrices are invertible

Add second row to third

e.g.:

$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =: A$$

Elem. matrix

Subtract second row from third

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} =: B$$

Exercise: Check that $AB = BA = I$.

Definition:

A matrix is a column-padded identity if it's obtained from the identity by adding in columns, but never putting in a nonzero entry to the left of a 1 from the original identity matrix.

e.g.:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

SKILL: Row magic on a nonzero matrix

1. Find leftmost nonzero column.
2. In that column, find topmost nonzero entry.
3. Interchange
   row with that entry
   and
   top row,
   thereby moving the nonzero entry
   to the top row.
   Call that nonzero entry in the top row
   the "pivot entry".
4. Divide the top row by the pivot entry, changing the pivot entry to 1.
5. For each row number $k \geq 2$, let
   $c_k :=$ entry in row k below the pivot entry,
   use the secondary row op R1$\times(-c_k)\to$Rk,
   to change c_k to zero.

SKILL: Row magic on a nonzero matrix

$$\begin{bmatrix} 0 & 0 & 3 & 6 & -21 \\ -5 & 0 & -10 & -15 & 35 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} -5 & 0 & -10 & -15 & 35 \\ 0 & 0 & 3 & 6 & -21 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix}$$

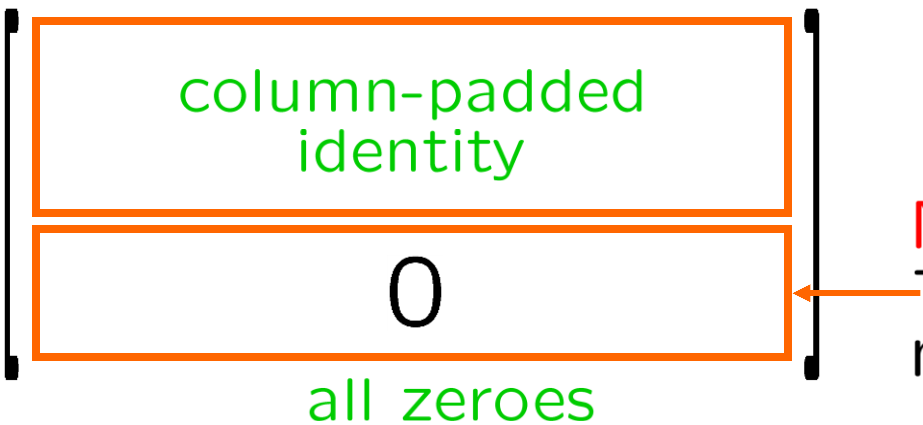
$R1 \times (-1/5) \downarrow$

$$\begin{bmatrix} 1 & 0 & 2 & 3 & -7 \\ 0 & 0 & 3 & 6 & -21 \\ 0 & 0 & 4 & 6 & -18 \\ 0 & 0 & -1 & 1 & -8 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 & 0 & 2 & 3 & -7 \\ 0 & 0 & 3 & 6 & -21 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix}$$

$R1 \times 4 \to R3$
$R1 \times 3 \to R4$

SKILL: Row canonical form

Do row magic.
Submatrix := everything except first row.
If submatrix is all zeroes, stop; otherwise:
Work the magic in the submatrix.
Use the submatrix pivot entry to
clear all entries in the matrix above it.
Submatrix := everything except first two rows.
Repeat until we hit:



NOTE:
This matrix of zeroes
may be empty.

SKILL: Row canonical form

R2×(1/3)

$$\begin{bmatrix} 1 & 0 & 2 & 3 & -7 \\ 0 & 0 & 3 & 6 & -21 \\ 0 & 0 & 4 & 6 & -18 \\ 0 & 0 & -1 & 1 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & -7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 4 & 6 & -18 \\ 0 & 0 & -1 & 1 & -8 \end{bmatrix}$$

R2×(−4) →R3

R2×1 →R4

$$\begin{bmatrix} 1 & 0 & 2 & 3 & -7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix}$$

R2×(−2) →R1

column-padded identity

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix}$$

not all zeroes, so continue

SKILL: Row canonical form

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix}$$

SKILL: Row canonical form

$R3\times(-1/2)$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix}$$

$R3\times(-3)\rightarrow R4$ ↓

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R3\times 1\rightarrow R1$
$R3\times(-2)\rightarrow R2$

←

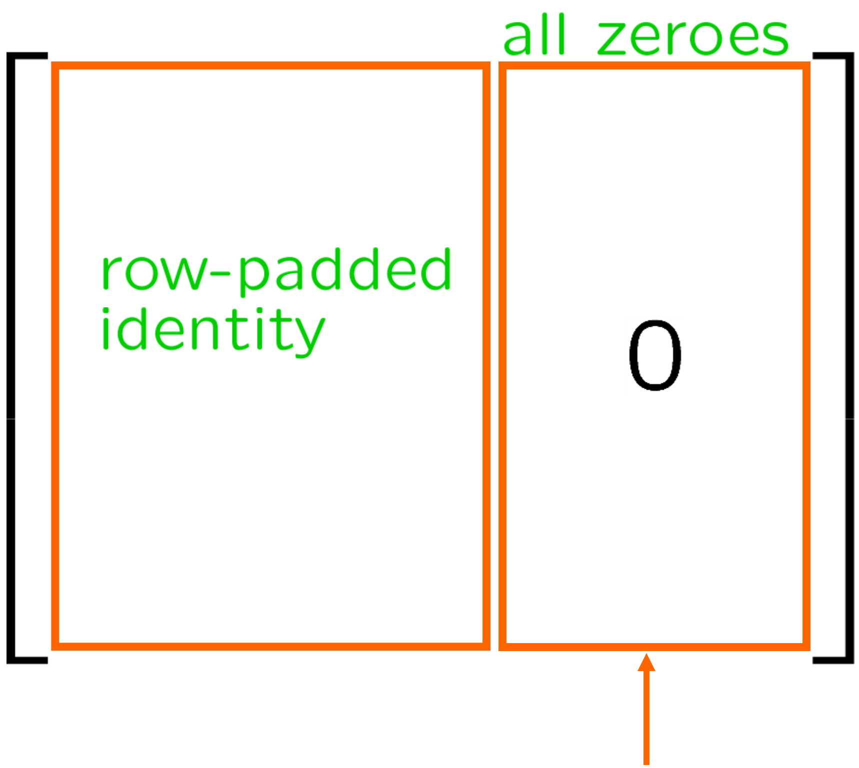
column-padded identity

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all zeroes, so stop

Exercise: Define row-padded identity, define column magic and column canonical form.

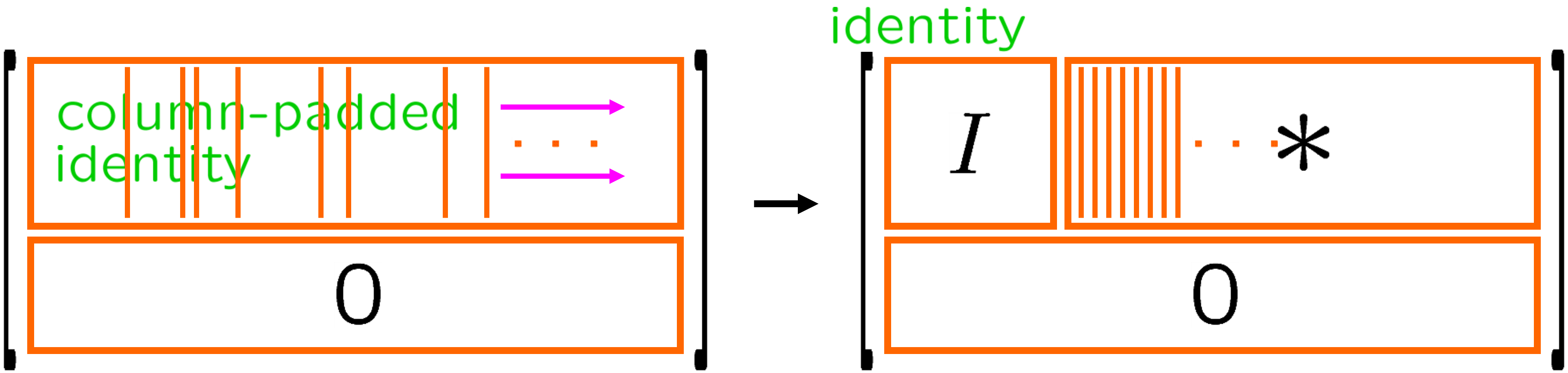
column canonical form:



NOTE: This matrix of zeroes may be empty.

SKILL: Fully canonical form

Do row canonical form.



By interchanging columns,
move the added columns in the column-padded identity
to the right, yielding
an identity matrix in the upper left,
and another matrix to its right.

SKILL: Fully canonical form

identity

$$\left[\begin{array}{c|c} I & 0 \\ \hline \multicolumn{2}{c}{0} \end{array}\right] \leftarrow \left[\begin{array}{c|c} I & * \\ \hline \multicolumn{2}{c}{0} \end{array}\right]$$

Using the ones in the identity matrix,
clear all entries in the arbitrary $(*)$ matrix.

Fully canonical form

Identity in the upper left
and zeroes elsewhere.

SKILL: Fully canonical form

column-padded identity

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all zeroes

$\rightarrow$ $C2 \leftrightarrow C3$, $C3 \leftrightarrow C4$

identity | arbitrary

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all zeroes

$\downarrow$

$C1 \times (-2) \rightarrow C5$

$C2 \times (-3) \rightarrow C5$

$C3 \times 5 \rightarrow C5$

identity | all zeroes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all zeroes

Fully canonical form

identity | all zeroes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

all zeroes | all zeroes

Fully canonical form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem:

Any matrix M can be written

$$M = E_1 \cdots E_k C E'_1 \cdots E'_l$$

where $E_1, \ldots, E_k, E'_1, \ldots, E'_l$ are elementary, and C is fully canonical.

e.g.:

$$M := \begin{bmatrix} 0 & 0 & 3 & 6 & -21 \\ -5 & 0 & -10 & -15 & 35 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix}$$

↓

12 elementary row operations ⋮

↓

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E''_1 \cdots E''_{12} M$$

e.g.:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E''_1 \cdots E''_{12} M$$

$\downarrow$ ⋮ 5 elementary column operations $\downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E''_1 \cdots E''_{12} M$$

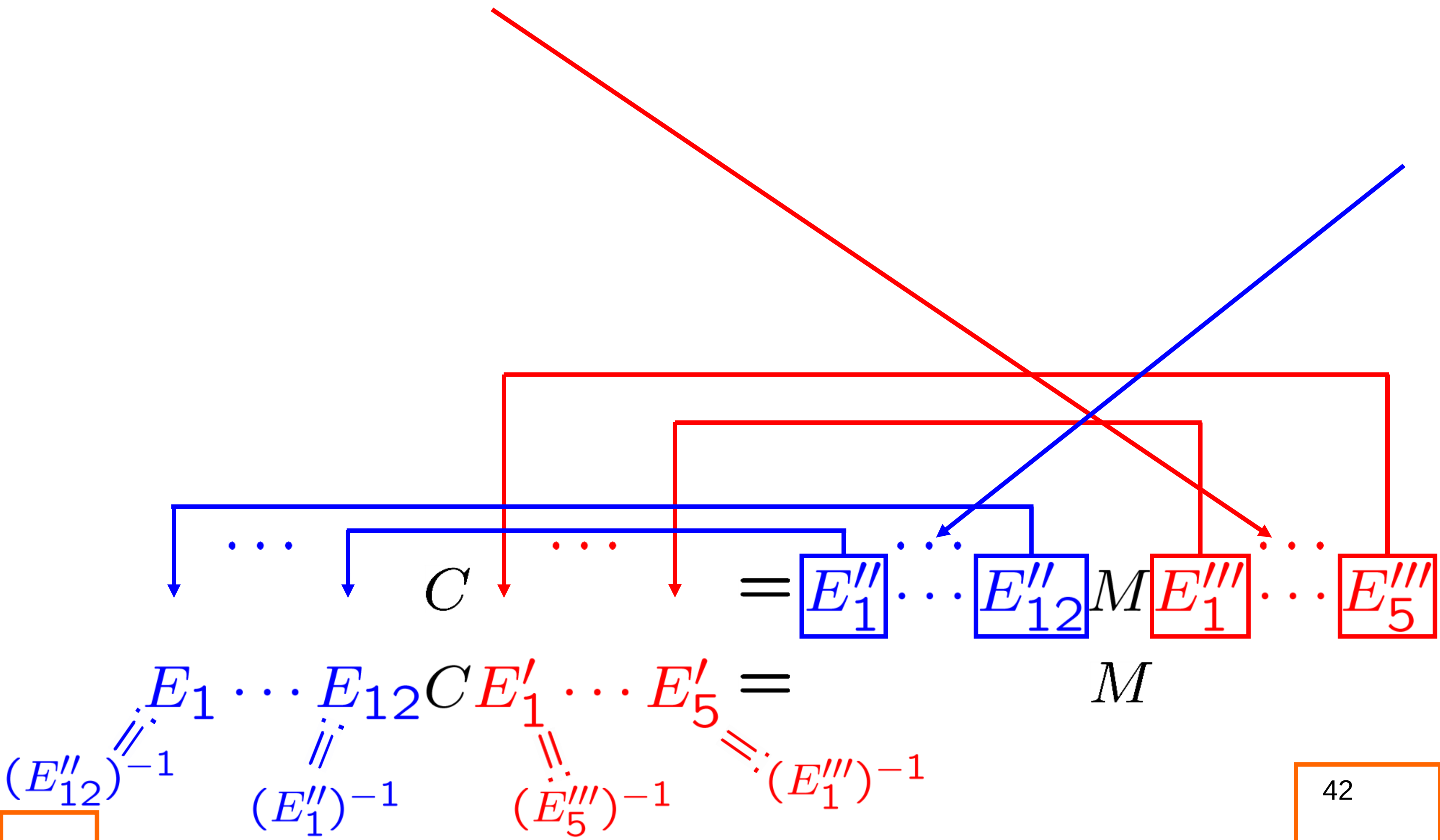
e.g.:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E''_1 \cdots E''_{12} M$$

$\downarrow$ $\vdots$ 5 elementary column operations $\downarrow$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E''_1 \cdots E''_{12} M E'''_1 \cdots E'''_5$$

$$C = E''_1 \cdots E''_{12} M E'''_1 \cdots E'''_5$$



$\cdots \quad \cdots \quad C \quad \cdots = E''_1 \cdots E''_{12} M E'''_1 \cdots E'''_5$
$E_1 \cdots E_{12} C E'_1 \cdots E'_5 = M$
$(E''_{12})^{-1}$
$(E''_1)^{-1}$
$(E'''_5)^{-1}$
$(E'''_1)^{-1}$

Theorem:

Any matrix M can be written

$$M = E_1 \cdots E_k C E'_1 \cdots E'_l$$

where $E_1, \ldots, E_k, E'_1, \ldots, E'_l$ are elementary, and C is fully canonical.

e.g.:

$$M := \begin{bmatrix} 0 & 0 & 3 & 6 & -21 \\ -5 & 0 & -10 & -15 & 35 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise: Write down $E_1, \ldots, E_{12}, E'_1, \ldots, E'_5$.

$$C = E''_1 \cdots E''_{12} M E'''_1 \cdots E'''_5$$

$$E_1 \cdots E_{12} C E'_1 \cdots E'_5 = M$$

$E_1 = (E''_{12})^{-1}$, $E_{12} = (E''_1)^{-1}$, $E'_1 = (E'''_5)^{-1}$, $E'_5 = (E'''_1)^{-1}$

SKILL:

Given a matrix M, write it as a product

$$M = E_1 \cdots E_k C E'_1 \cdots E'_l$$

where $E_1, \ldots, E_k, E'_1, \ldots, E'_l$ are elementary, and C is fully canonical.

STOP

SKILL:

Given n and k, write down all the fully canonical $n \times k$ matrices.

e.g.: List the fully canonical 4×3 matrices.

Solution:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$