Financial Mathematics

Determinants exist

Definition:

An **oriented parallelogram** is
an ordered pair of ordered pairs of scalars.

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e.g.:

$$(\quad (1,3) \quad , \quad (4,2) \quad)$$

Questions:

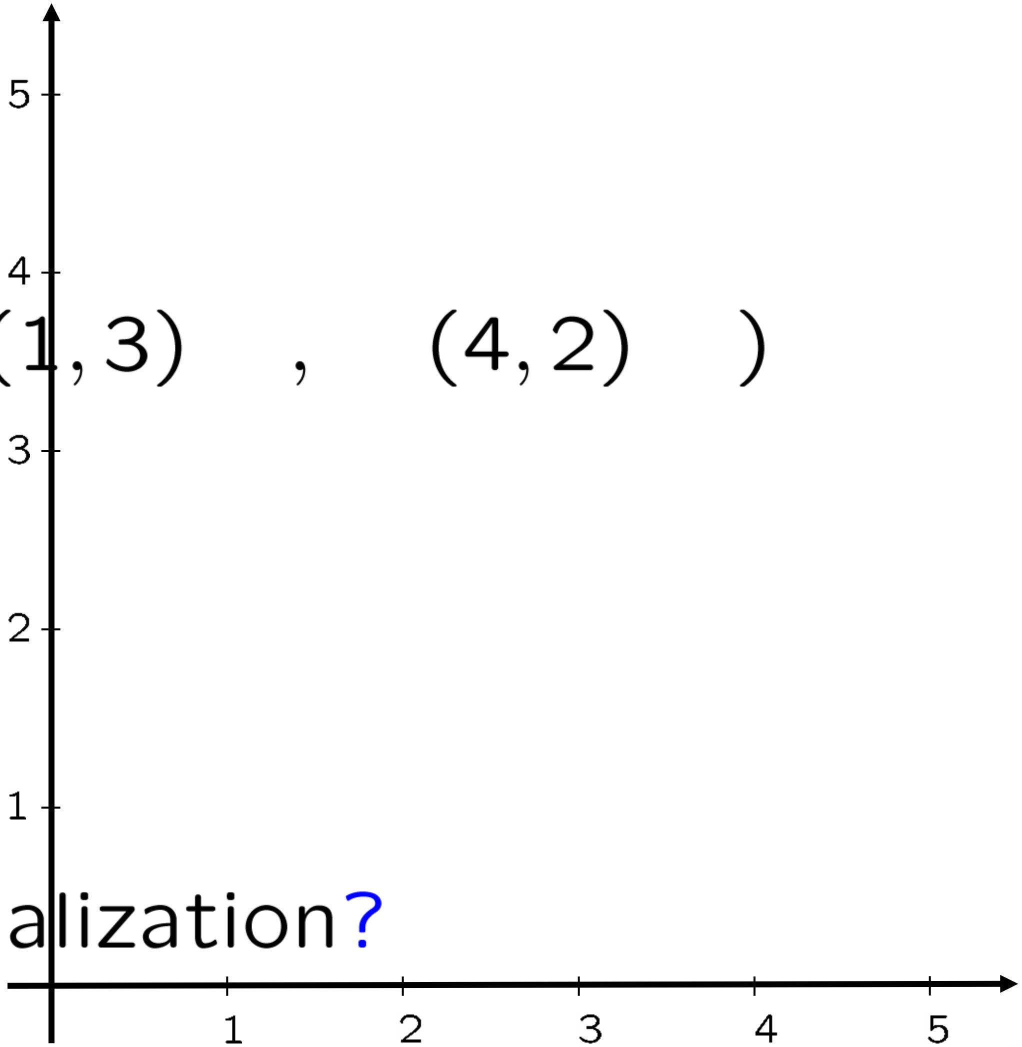
Why call this a kind of parallelogram?

Visualization?

e.g.:

$$\left(\quad (1,3) \quad , \quad (4,2) \quad \right)$$

Visualization:



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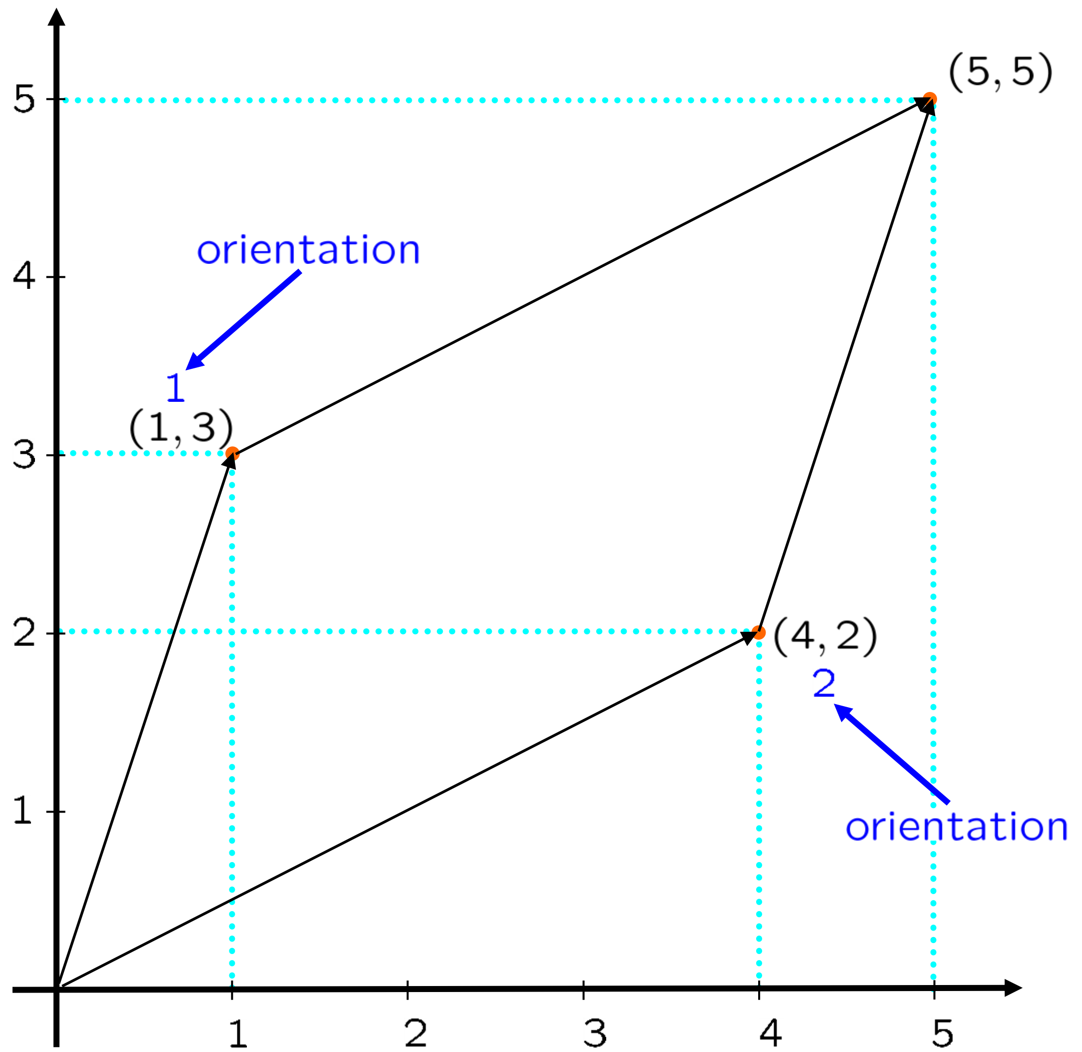


Definition:

An **oriented (3-dimensional) parallelpiped** is an ordered triples of ordered triples of scalars.

---

e.g.:

$$(\quad (1,2,3) \quad , \quad (4,5,6) \quad , \quad (7,8,9) \quad)$$

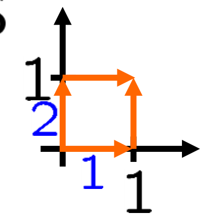
Exercise: Visualization?

(Remember the orientation – the blue numbers.)

Definition:
An **oriented parallelogram** is
an ordered pair of ordered pairs of scalars.

Definition:
The **standard oriented parallelogram** is

$$\Big(\quad (1,0) \quad , \quad (0,1) \quad \Big)$$



Definition:
An **oriented (3-dimensional) parallelpiped** is
an ordered triples of ordered triples.

Definition:
The **standard oriented 3-parallelpiped** is

$$\Big(\quad (1,0,0) \quad , \quad (0,1,0) \quad , \quad (0,0,1) \quad \Big)$$

IMAGINE!!

Definition:

An **oriented (n-dimensional) parallelpiped**, or **oriented n-parallelpiped**, is an ordered $n$tuple of ordered $n$tuples of scalars.

Definition:

The **standard oriented n-parallelpiped** is

$$(\quad (1,0,\ldots,0) \quad , \quad (0,1,0,\ldots,0) \quad , \quad \ldots \quad , \quad (0,\ldots,0,1) \quad)$$

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$$(\quad (1,0,0) \quad , \quad (0,1,0) \quad , \quad (0,0,1) \quad)$$

Definition:

An oriented n-parallelpiped is **degenerate** if it has no n-dimensional volume.

---

e.g.:

An oriented parallelogram is **degenerate** if it has no area.

---

e.g.: $(\quad (1,2) \quad , \quad (1,2) \quad)$

$(\quad (1,2) \quad , \quad (2,4) \quad)$

$(\quad (1,2) \quad , (-2,-4) \ )$

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---

e.g.:

An oriented 3-parallelpiped is **degenerate** if it has no (3-dimensional) volume.

---

e.g.: ($(1,2,3)$, $(0,3,2)$, $(2,7,8)$)

Note: $2 \cdot (1,2,3) + (0,3,2) = (2,7,8)$

Definition:

An oriented n-parallelpiped is **positive**
if there's a continuous path
from it
to the standard oriented n-parallepiped
which is never degenerate.

---

non-*e.g.*:

$(\quad (1,3) \quad , \quad (4,2) \quad)$

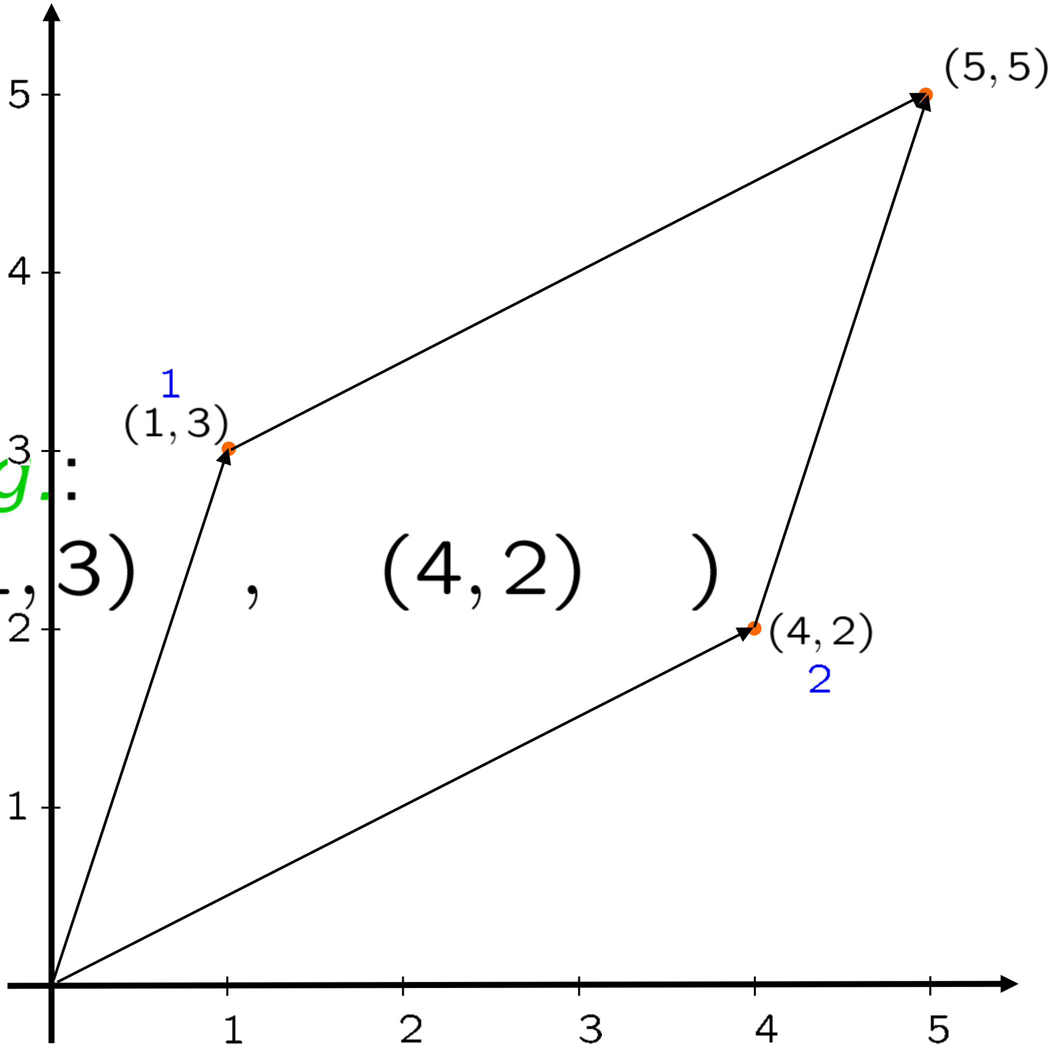
Question:

Why is $(\quad (1,3) \quad , \quad (4,2) \quad)$ not positive?
Visualization?

non-*e.g.*:

$$\left(\quad (1,3) \quad , \quad (4,2) \quad \right)$$

Visualization:



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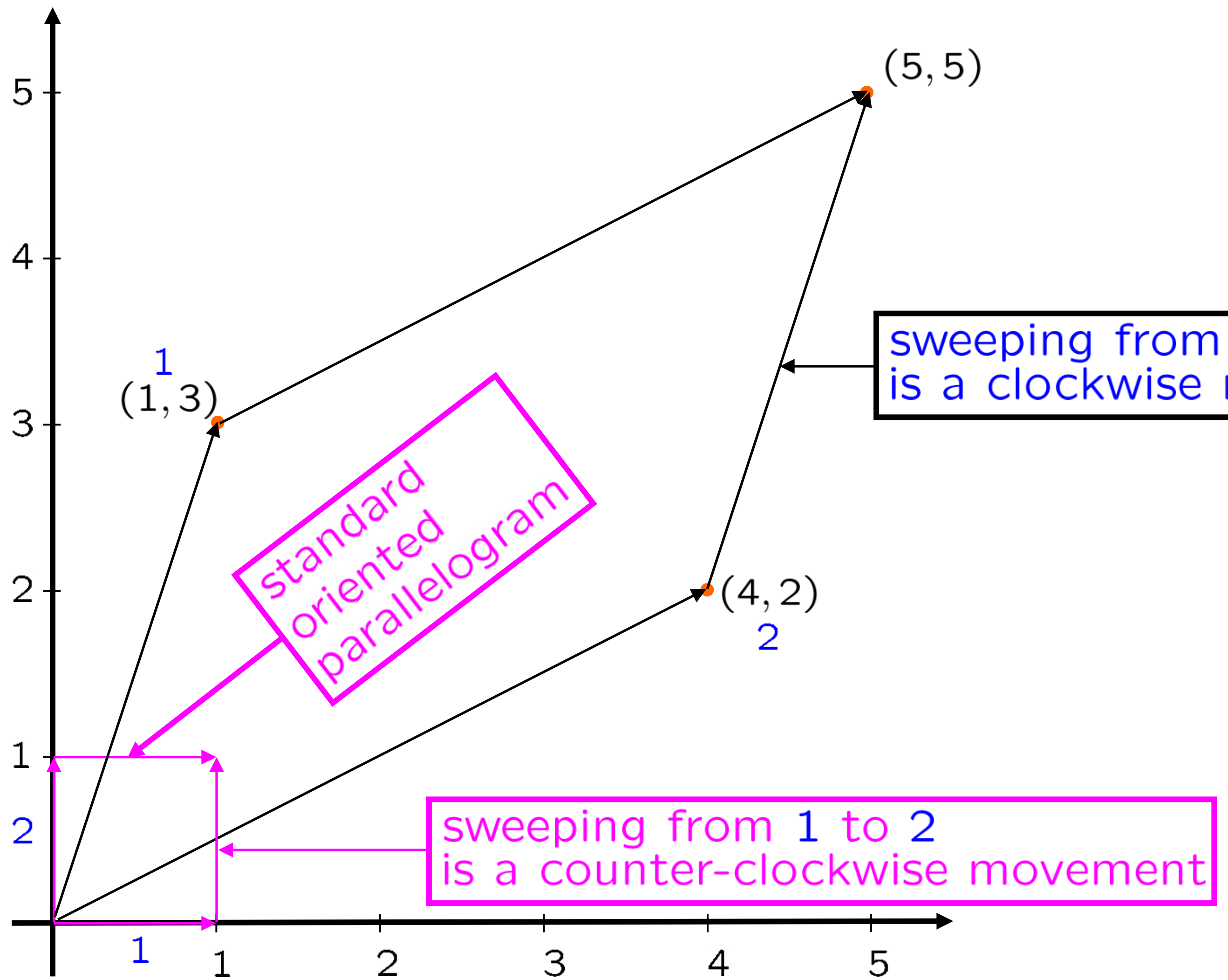

Definition:
An oriented n-parallelpiped is **negative** if it's neither positive nor degenerate.

Definition:
The **signed n-volume** of an oriented n-parallelpiped of n-volume A is:

A if the parallelpiped is positive;
0 if the parallelpiped is degenerate;
$-A$ if the parallelpiped is negative;

Note: The **signed 2-volume** of an oriented parallelogram is called its **signed area**.

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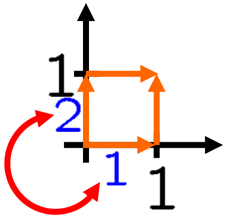
e.g.: The signed area of the standard parallelogram

$$(\quad (1,0) \quad , \quad (0,1) \quad)$$

is: 1. -1



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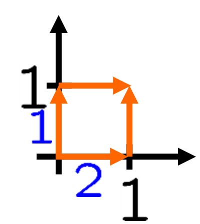

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IOU

Exercise: Find the signed area of

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Exercise: Find the signed area of

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Notation: For any oriented n-parallelpiped P, the signed n-volume of P is denoted $\boxed{\text{sv}(P)}$

Definition:

For any $n \times n$ matrix A,
for any oriented n-parallelpiped

$$P = (v_1, \ldots, v_n), \qquad (v_1, \ldots, v_n \in \mathbb{R}^n)$$

we define:

$$\boxed{AP} = (L_A(v_1), \ldots, L_A(v_n)).$$

---

e.g.:

Let $A := \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Then $L_A(x, y) = (x + 4y, 3x + 2y)$.

Then $L_A(1, 0) = (1, 3)$

and $L_A(0, 1) = (4, 2)$,

so $A(\quad (1,0) \quad , \quad (0,1) \quad)$

$= (\quad (1,3) \quad , \quad (4,2) \quad)$.

Definition:

Let A be an $n \times n$ matrix.

We say that A **has a determinant** if there is a number, denoted $\boxed{\det(A)}$, such that, for *ANY* oriented n-parallelpiped P,

$$\mathrm{sv}(AP) = [\det(A)][\mathrm{sv}(P)].$$

---

e.g.: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles base, fixes height, so has a determinant.

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2.$$

(reducing parentheses)

Note: One usually writes: $\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

e.g.: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ has a determinant.

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = 3.$$

Fact: Let $A, B \in \mathbb{R}^{n \times n}$.

If A and B both have determinants,
then AB has a determinant,
and $\det(AB) = [\det(A)][\det(B)]$.

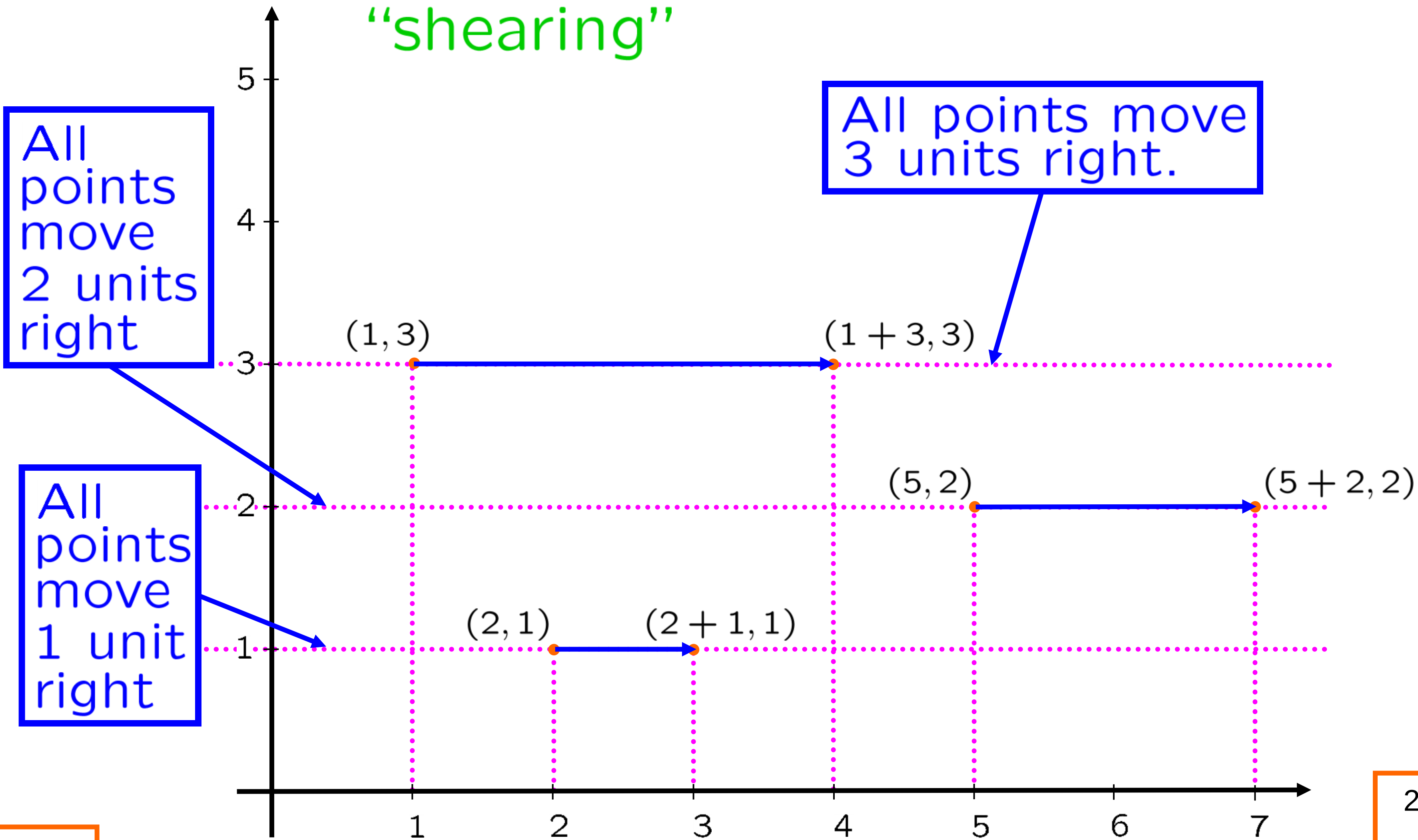
---

e.g.: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ has a determinant.

$$\det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 2 \cdot 3 = 6.$$

Fact: Any diagonal matrix has a det., equal to the product of all diagonal entries.

Let $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad L_A(x, y) = (x + y, y)$



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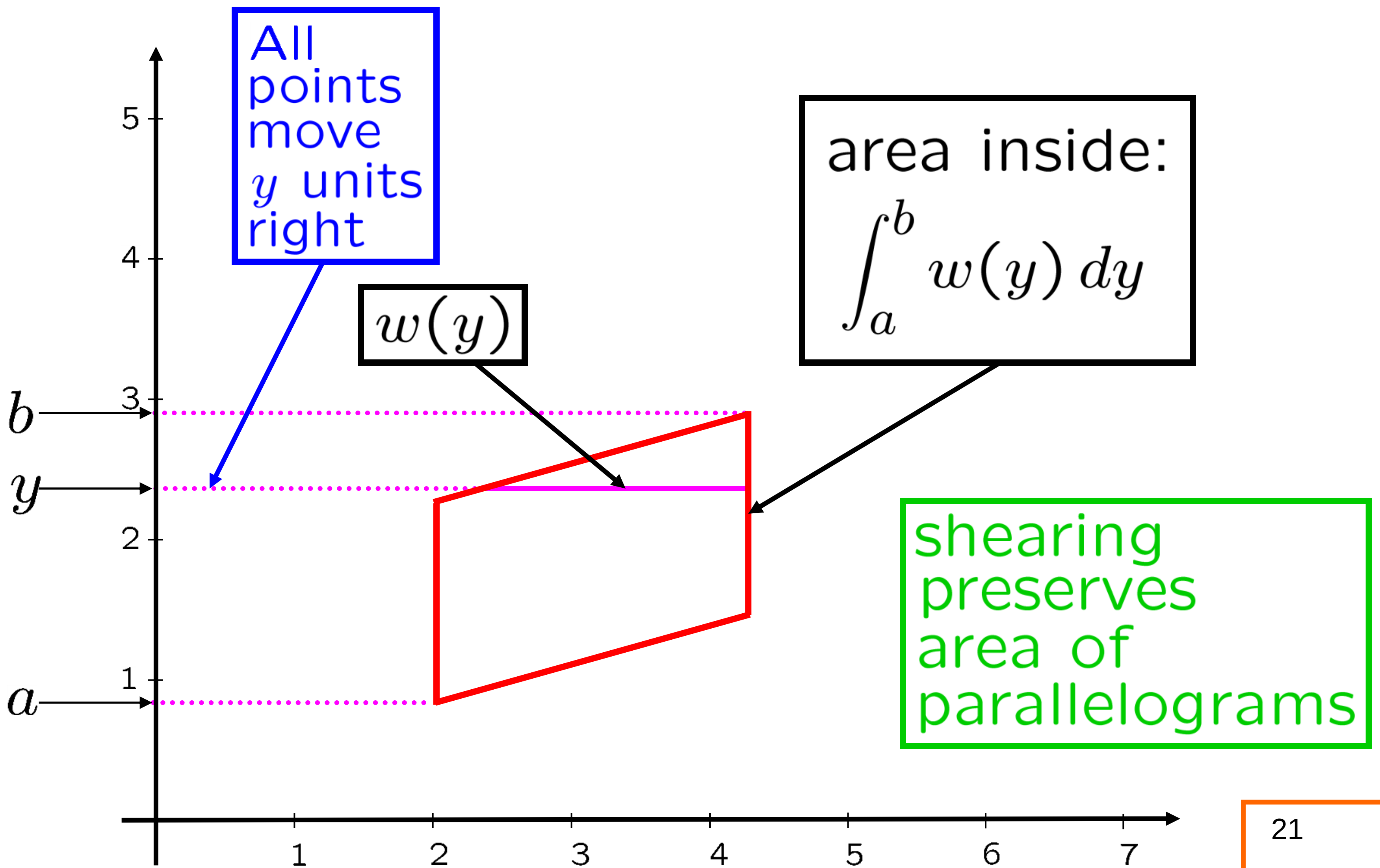

Let $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad L_A(x,y) = (x+y, y)$

For all oriented parallelograms P,
we have: $\mathrm{sv}(P) = \mathrm{sv}(AP)$.

A has a determinant, and $\det(A) = 1$.

shearing preserves signed area of oriented parallelograms

shearing preserves area of parallelograms *e.g.*: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has a determinant.

$$\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1.$$

Fact: A square matrix which agrees with the identity except in one non-diagonal entry has a determinant, with $\det = 1$.

Fact: Any elementary matrix has a determinant.

Theorem:

Any matrix M can be written

$$M = E_1 \cdots E_k C E'_1 \cdots E'_l$$

where $E_1, \ldots, E_k, E'_1, \ldots, E'_l$ are elementary, and C is fully canonical.

e.g.:

$$M := \begin{bmatrix} 0 & 0 & 3 & 6 & -21 \\ -5 & 0 & -10 & -15 & 35 \\ -4 & 0 & -4 & -6 & 10 \\ -3 & 0 & -7 & -8 & 13 \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_1 \cdots E_{12} C E'_1 \cdots E'_5 = M$$

Note: A square canonical matrix is diagonal, and so has a determinant.

Theorem:

Every square matrix has a determinant.

Fact:

Let A be an $n \times n$ matrix.
Then there is a number, denoted $\det(A)$,
such that, for *ANY* oriented n-parallelpiped P,

$$\text{sv}(AP) = [\det(A)][\text{sv}(P)].$$

Corollary:

Let A be an $n \times n$ matrix.
Let P be an n-dim'l parallelpiped.
(*e.g.*, an n-dim'l rectangular box).
Then $\text{vol}(L_A(P)) = [|\det(A)|][\text{vol}(P)]$.

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Let A be an 2×2 matrix, let $p, q > 0$.
Then $\mathrm{Area}(L_A([0,p] \times [0,q]))$

$$= [|\det(A)|][\mathrm{Area}(0,p] \times [0,q])]$$

$$= [|\det(A)|]pq$$

Let A be an 2×2 matrix, let $a, b, c, d > 0$.

$$(L_A(-a,-c)) + (L_A([0, a+b] \times [0, c+d]))$$

Then $\text{Area}(L_A([-a, b] \times [-c, d]))$

$$(-a,-c) + ([0, a+b] \times [0, c+d])$$

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$$= [|\det(A)|](a+b)(c+d)$$

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$p :\to a+b,\ q :\to c+d$

e.g.: Let $A := \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. $\det(A) =$??

$$E_1 := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad E_1 A = \begin{bmatrix} 1 & 4 \\ 0 & -10 \end{bmatrix}$$

$$E_2 := \begin{bmatrix} 1 & 0 \\ 0 & -1/10 \end{bmatrix} \qquad E_2 E_1 A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$[-1/10][1][\det(A)] = \det(E_2 E_1 A) = 1$$

$$\det(A) = -10$$

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Exercise: Find the signed area of

$$(\quad (1,3) \quad , \quad (4,2) \quad).$$

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Exercise: Find the signed area of

$$(\quad (1,3) \quad , \quad (4,2) \quad).$$

Solution: $P := (\quad (1,0) \quad , \quad (0,1) \quad).$

$$AP = (\quad (1,3) \quad , \quad (4,2) \quad).$$

Want: $\text{sv}(AP)$

$$\text{sv}(AP) = [\det(A)][\text{sv}(P)]$$

$$= [-10][1] = -10$$