Financial Mathematics

Properties of determinants

$E_1, \ldots, E_k, E'_1, \ldots, E'_l$ primary elementary

Say $M = E_1 \cdots E_k C E'_1 \cdots E'_l$. Then

$$\det(M) = [\det(E_1)] \cdots [\det(E_k)] \times [\det(C)][\det(E'_1)] \cdots [\det(E'_l)]$$

Also, $M^t = (E'_l)^t \cdots (E'_1)^t C^t (E_k)^t \cdots (E_1)^t$.

Then

$$\det(M^t) = [\det(E'_l)^t] \cdots [\det(E'_1)^t] \times [\det(C^t)][\det(E_k)^t] \cdots [\det(E_1)^t]$$

Also, for all integers $j \in [1, l]$,

$$\det(E'_j) = \det((E'_j)^t)$$

Also, $C = C^t$.

Also, for all integers $j \in [1, k]$,

$$\det(E_j) = \det(E_j^t)$$

Then $\det(M) = \det(M^t)$.

$\det(C \oplus D)$, with C, D diagonal

Fact: $\det(C \oplus D) = (\det C)(\det D)$, $\forall$diag. C, D

e.g.:

$$\left(\det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)\left(\det \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

$$\|$$

$$\det\Bigg(\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}}\Bigg) = (2)(3)(4)(5)(6)$$

det$(C \oplus D)$, with C, D diagonal

Fact: $\det(C \oplus D) = (\det C)(\det D)$, $\forall$diag. C, D

Cor: $\det(I \oplus E) = \det E$, $\forall$id. I, pr. elem. E

$\det(E \oplus I) = \det E$, $\forall$id. I, pr. elem. E

$\det(L \oplus M)$

Let $L \in \mathbb{R}^{p\times p}$, $M \in \mathbb{R}^{q\times q}$.

Obtain a diagonal matrix C from L, via primary row and col. ops, w/ $\prod$ dets $= a$.

$a \cdot (\det L) = \det C$ $\quad \bullet \oplus I$

Obtain a diagonal matrix D from M, via primary row and col. ops, w/ $\prod$ dets $= b$.

$b \cdot (\det M) = \det D$ $\quad I \oplus \bullet$

$$ab \cdot [\det(L \oplus M)] = \det(C \oplus D)$$

$$L \oplus M = \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}$$

primary row and col. ops, w/ $\prod$ dets $= a$

$$\begin{bmatrix} C & 0 \\ 0 & M \end{bmatrix} \longrightarrow \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} = C \oplus D$$

primary row and col. ops, w/ $\prod$ dets $= b$

$\det(L \oplus M)$

Let $L \in \mathbb{R}^{p\times p}$, $M \in \mathbb{R}^{q\times q}$.

Obtain a diagonal matrix C from L, via primary row and col. ops, w/ $\prod$ dets $= a$.

$a \cdot (\det L) = \det C$

Obtain a diagonal matrix D from M, via primary row and col. ops, w/ $\prod$ dets $= b$.

$b \cdot (\det M) = \det D$

$$ab \cdot [\det(L \oplus M)] = \det(C \oplus D)$$

$$\begin{aligned} ab \cdot [\det(L \oplus M)] &= \det(C \oplus D) \\ &= [\det C][\det D] \\ &= [a \cdot (\det L)][b \cdot (\det M)] \end{aligned}$$

$\det(L \oplus M)$

Let $L \in \mathbb{R}^{p \times p}$, $M \in \mathbb{R}^{q \times q}$.

Obtain a diagonal matrix C from L, via primary row and col. ops, w/ $\prod$ dets $= a$.

$a \cdot (\det L) = \det C$

Obtain a diagonal matrix D from M, via primary row and col. ops, w/ $\prod$ dets $= b$.

$b \cdot (\det M) = \det D$

$$\cancel{a}\cancel{b} \cdot [\det(L \oplus M)] = [\cancel{a} \cdot (\det L)][\cancel{b} \cdot (\det M)]$$

$$\begin{aligned} \det(L \oplus M) &= (\det L)(\det M) \\ &= [a \cdot (\det L)][b \cdot (\det M)] \end{aligned}$$

$\det(L \oplus M)$

Let $L \in \mathbb{R}^{p\times p}$, $M \in \mathbb{R}^{q\times q}$.

Obtain a diagonal matrix C from L, via primary row and col. ops, w/ $\prod$ dets $= a$.

$a \cdot (\det L) = \det C$

Obtain a diagonal matrix D from M, via primary row and col. ops, w/ $\prod$ dets $= b$.

$b \cdot (\det M) = \det D$

The determinant of the direct sum is the *PRODUCT* of the determinants.

$$\det(L \oplus M) = (\det L)(\det M)$$

Next topic: Determinant is alternating and multilinear

Notation: If $v = (a,b)$ and $w = (c,d)$,

then $[v \quad w]$ means $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Then: $A = [v \quad w]$ implies that

$L_A(1,0) = v$ and $L_A(0,1) = w$,

so $A((1,0),(0,1)) = (v,w)$,

so $[\det A][\underbrace{\mathrm{sv}((1,0),(0,1))}_{1}] = \mathrm{sv}(v,w)$,

so $\mathrm{sv}(v,w) = \det(A)$.

Then: $\mathrm{sv}(v,w) = \det[v \quad w]$.

$$\det[v \quad w] = \mathsf{sv}(v, w)$$
$$= -\mathsf{sv}(w, v)$$

$$\mathsf{sv}(v, w) = \det[v \quad w]$$

$$\begin{aligned}\det[v \quad w] &= \mathsf{sv}(v, w) \\ &= -\mathsf{sv}(w, v) \\ &= -\det[w \quad v]\end{aligned}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

"The determinant is alternating in columns."

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

transpose matrices on left and right hand sides

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

transpose matrices on left and right hand sides

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\det \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

"The determinant is alternating in rows."

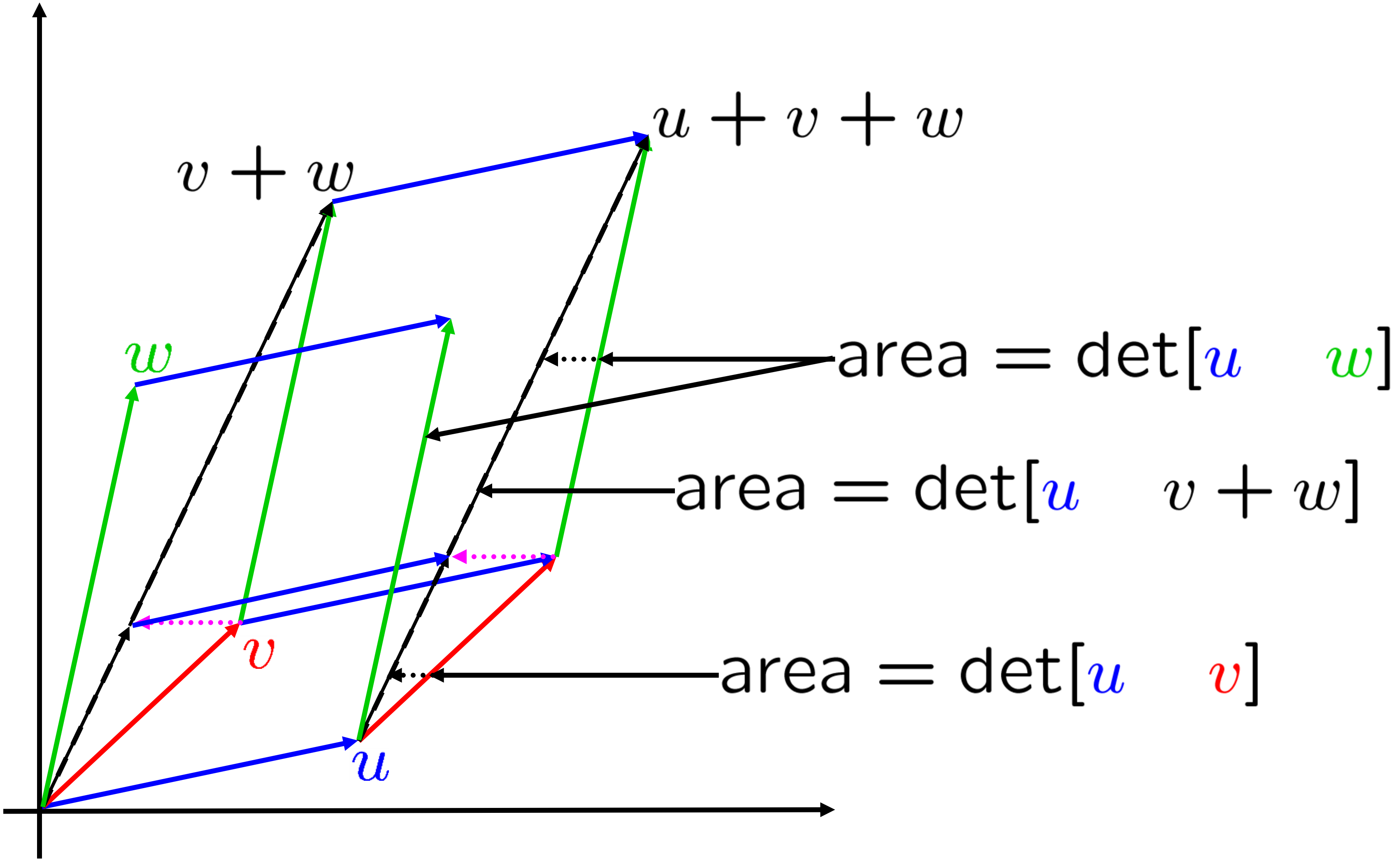
$$\det \begin{bmatrix} a & b \\ a & b \end{bmatrix} = -\det \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \text{ so } \det \begin{bmatrix} a & b \\ a & b \end{bmatrix} = 0$$

$$x = -x \quad \Rightarrow \quad x = 0$$

"If two rows are equal, then $\det = 0$.

Transpose:
"If two columns are equal, then $\det = 0$.

Next topic: multilinearity of determinant



$$\det[u \quad v] + \det[u \quad w] = \det[u \quad v + w]$$

$$\det[u \quad v] + \det[u \quad w] = \det[u \quad v + w]$$

interchange first and second columns all through

$$\det[u \quad v] + \det[u \quad w] = \det[u \quad v + w]$$

$$\det[u \quad v] + \det[u \quad w] = \det[u \quad v+w]$$

interchange first and second columns all through

$$-\det[v \quad u] - \det[w \quad u] = -\det[v+w \quad u]$$

multiply by minus one

$$\det[v \quad u] + \det[w \quad u] = \det[v+w \quad u]$$

"The determinant is additive in columns."

$$\det \begin{bmatrix} a + 4a' + 7a'' & b \\ c + 4c' + 7c'' & d \end{bmatrix}$$

$$= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ \det \begin{bmatrix} 4a' & b \\ 4c' & d \end{bmatrix}$$

$$+ \det \begin{bmatrix} 7a'' & b \\ 7c'' & d \end{bmatrix}$$

$$\det \begin{bmatrix} a + 4a' + 7a'' & b \\ c + 4c' + 7c'' & d \end{bmatrix}$$

$$= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ \det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \det \begin{bmatrix} a'' & b \\ c'' & d \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} a + 4a' + 7a'' & b \\ c + 4c' + 7c'' & d \end{bmatrix}$$

$$= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ \det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix} \cdot \det \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \det \begin{bmatrix} a'' & b \\ c'' & d \end{bmatrix} \cdot \det \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} a + 4a' + 7a'' & b \\ c + 4c' + 7c'' & d \end{bmatrix}$$

$$= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ 4 \cdot \det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix}$$

$$+ 7 \cdot \det \begin{bmatrix} a'' & b \\ c'' & d \end{bmatrix}$$

"The determinant is multilinear in columns."

$$\det \begin{bmatrix} a + 4a' + 7a'' & b & c \\ d + 4d' + 7d'' & e & f \\ g + 4g' + 7g'' & h & i \end{bmatrix}$$

$$= \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+ \; 4 \cdot \det \begin{bmatrix} a' & b & c \\ d' & e & f \\ g' & h & i \end{bmatrix}$$

$$+ \, 7 \cdot \det \begin{bmatrix} a'' & b & c \\ d'' & e & f \\ g'' & h & i \end{bmatrix}$$

$$\det \begin{bmatrix} a+a' & b \\ c+c' & d \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix}$$

transpose matrices on left and right hand sides

$$\det \begin{bmatrix} a+a' & c+c' \\ b & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \det \begin{bmatrix} a' & c' \\ b & d \end{bmatrix}$$

"The determinant is additive in rows."

"The determinant is multilinear in rows."

If a row is zero, det is zero

$$\det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0+0 & 0+0 & 0+0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

If a row is zero, det is zero

$$\det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

If a row is zero, det is zero

$$0$$

$$\| \qquad \backslash\backslash$$

$$0 \qquad + \qquad \det \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{bmatrix}$$

If a column is zero, det is zero

$$\det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0+0 & b & c \\ 0+0 & e & f \\ 0+0 & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} \quad + \quad \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

If a column is zero, det is zero

$$\det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

$$\|$$

$$\det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

If a column is zero, det is zero

$$0$$

$$\| \qquad \backslash\backslash$$

$$0 \qquad + \qquad \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$$

Next topic: determinant formulas

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ??$$

$$A = \begin{bmatrix} a + 0 & b \\ 0 + c & d \end{bmatrix}$$

$$\det(A) = \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

$$\det(A) = \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - \det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ??$$

$$\det(A) = \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - \det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$$\det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$\det(A) = \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} - \det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ??$$

$$\det(A) = \det \overbrace{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}^{ad} - \det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$$\det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \det \begin{bmatrix} a & b+0 \\ 0 & 0+d \end{bmatrix}$$

$$= \det \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = ad$$

$$\det \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = 0 \qquad \det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = ad$$

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ??$$

$$\det(A) = \overbrace{\det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}^{ad} - \overbrace{\det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}}^{bc}$$

$$\det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = cb = bc$$

$$\det(A) = ad - bc$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

3×3 case?

NOTE: $2! = 2$ monomials, half with $+$, half with $-$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$\det \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & b & c \\ d & e & f \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{bmatrix}$$

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + 0 + 0$$

$$\det \begin{bmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & h & i \end{bmatrix} + 0 + 0$$

$$\det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & 0 & 0 \end{bmatrix} + 0 + 0$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & 0 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix} - \det \begin{bmatrix} d & 0 & 0 \\ 0 & b & c \\ 0 & h & i \end{bmatrix} \quad \det \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & 0 & 0 \end{bmatrix}$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\det\begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix} + \underbrace{\det\begin{bmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & h & i \end{bmatrix}}_{-\det\begin{bmatrix} d & 0 & 0 \\ 0 & b & c \\ 0 & h & i \end{bmatrix}} + \underbrace{\det\begin{bmatrix} 0 & b & c \\ 0 & e & f \\ g & 0 & 0 \end{bmatrix}}_{\det\begin{bmatrix} g & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{bmatrix}}$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix} - \det \begin{bmatrix} d & 0 & 0 \\ 0 & b & c \\ 0 & h & i \end{bmatrix} + \det \begin{bmatrix} g & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{bmatrix}$$

$$- \det \begin{bmatrix} d & 0 & 0 \\ 0 & b & c \\ 0 & h & i \end{bmatrix} \quad \det \begin{bmatrix} g & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{bmatrix}$$

det of direct sum = product of dets

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\underbrace{\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix}}_{a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}} - \underbrace{\det \begin{bmatrix} d & 0 & 0 \\ 0 & b & c \\ 0 & h & i \end{bmatrix}}_{d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix}} + \underbrace{\det \begin{bmatrix} g & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{bmatrix}}_{g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}}$$

det of direct sum = product of dets

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix} + g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}$$

$$a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} \qquad d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix} \qquad g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}$$

Expanding along the first column

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$+\, a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix} + g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}$$

Note the alternating signs

(1,1) entry; 1+1 is even $\Rightarrow$ + sign
(2,1) entry; 2+1 is odd $\Rightarrow$ − sign
(3,1) entry; 3+1 is even $\Rightarrow$ + sign

Expanding along the first column

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$+ a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix} + g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}$$

$$= a(ei - hf) - d(bi - hc) + g(bf - ec)$$

$$= aei - ahf - dbi + dhc + gbf - gec$$

NOTE: 3!=6 monomials,
half with $+$, half with $-$

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} =$$

$$+a \det \begin{bmatrix} f & g & h \\ j & k & l \\ n & o & p \end{bmatrix} - e \det \begin{bmatrix} b & c & d \\ j & k & l \\ n & o & p \end{bmatrix}$$

$$+i \det \begin{bmatrix} b & c & d \\ f & g & h \\ n & o & p \end{bmatrix} - m \det \begin{bmatrix} b & c & d \\ f & g & h \\ j & k & l \end{bmatrix}$$

Expanding along the first column

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} =$$

NOTE: 4!=24 monomials, half with +, half with −

$$+a \det \begin{bmatrix} f & g & h \\ j & k & l \\ n & o & p \end{bmatrix} - b \det \begin{bmatrix} e & g & h \\ i & k & l \\ m & o & p \end{bmatrix}$$

$$+c \det \begin{bmatrix} e & f & h \\ i & j & l \\ m & n & p \end{bmatrix} - d \det \begin{bmatrix} e & f & g \\ i & j & k \\ m & n & o \end{bmatrix}$$

Expanding along the first row

Exercise: Work out formulas for expanding along the second row & third column.

$$\det \begin{bmatrix} a & b & c & d \\ 0 & f & g & h \\ 0 & 0 & k & l \\ 0 & 0 & 0 & p \end{bmatrix} = a \det \begin{bmatrix} f & g & h \\ 0 & k & l \\ 0 & 0 & p \end{bmatrix}$$

$$= af \det \begin{bmatrix} k & l \\ 0 & p \end{bmatrix}$$

$$= afkp$$

Expanding along the first column

Fact: The det. of an upper triangular matrix is the product of the diag. entries.

Fact: The det. of a lower triangular matrix is the product of the diag. entries.

Theorem: Let $M \in \mathbb{R}^{n \times n}$.
Then the following are equivalent:

- ($\bullet$) M is a product of elem. matrices
- ($\bullet$) M is reducible to I via elem. row & col. ops
- ($\bullet$) M is reducible to I via elem. row ops
- ($\bullet$) M is reducible to I via elem. col. ops
- ($\bullet$) M is invertible
- ($\bullet$) M has a left inverse
- ($\bullet$) M has a right inverse
- ($\bullet$) $L_M : \mathbb{R}^n \to \mathbb{R}^n$ is onto
- ($\bullet$) $\ker(M) = \{0\}$
- ($\bullet$) L_M is 1-1
- ($\bullet$) $\mathrm{im}(M) = \mathbb{R}^{n \times 1}$
- ($\bullet$) $\det(M) \neq 0$

Proof: True for diagonal matrices.
All ~~eleven~~ twelve properties are invariant under elementary row & column operations.

QED

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$+ \underbrace{a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}}_{a'} - \underbrace{b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}}_{b'} + \underbrace{c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}}_{c'}$$

Expanding along the first row

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a a' - b b' + c c' =$$

$$+a \underbrace{\det \begin{bmatrix} e & f \\ h & i \end{bmatrix}}_{a'} - b \underbrace{\det \begin{bmatrix} d & f \\ g & i \end{bmatrix}}_{b'} + c \underbrace{\det \begin{bmatrix} d & e \\ g & h \end{bmatrix}}_{c'}$$

$$0 = \det \begin{bmatrix} d & e & f \\ d & e & f \\ g & h & i \end{bmatrix} = d a' - e b' + f c'$$

$$0 = \det \begin{bmatrix} g & h & i \\ d & e & f \\ g & h & i \end{bmatrix} = g a' - h b' + i c'$$

$$D := \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a{\color{orange}a'} - b{\color{orange}b'} + c{\color{orange}c'}$$

$$0 = \det\begin{bmatrix} d & e & f \\ d & e & f \\ g & h & i \end{bmatrix} = d{\color{orange}a'} - e{\color{orange}b'} + f{\color{orange}c'}$$

$$0 = \begin{bmatrix} g & h & i \end{bmatrix} = g{\color{orange}a'} - h{\color{orange}b'} + i{\color{orange}c'}$$

$$0 = \det\begin{bmatrix} d & e & f \\ d & e & f \\ g & h & i \end{bmatrix} = d{\color{orange}a'} - e{\color{orange}b'} + f{\color{orange}c'}$$

$$0 = \det\begin{bmatrix} g & h & i \\ d & e & f \\ g & h & i \end{bmatrix} = g{\color{orange}a'} - h{\color{orange}b'} + i{\color{orange}c'}$$

$$D := \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aa' - bb' + cc'$$

$$0 = \det \begin{bmatrix} d & e & f \\ d & e & f \\ g & h & i \end{bmatrix} = da' - eb' + fc'$$

$$0 = \det \begin{bmatrix} g & h & i \\ d & e & f \\ g & h & i \end{bmatrix} = ga' - hb' + ic'$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' \\ -b' \\ c' \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' \\ -b' \\ c' \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -d' \\ e' \\ -f' \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' \\ -b' \\ c' \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' \\ -b' \\ c' \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} -d' \\ e' \\ -f' \end{bmatrix} = \begin{bmatrix} 0 \\ D \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} g' \\ -h' \\ i' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}$$

MATRIX OF MINORS

$$\begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{bmatrix}$$

COFACTOR MATRIX

$$\begin{bmatrix} a' & -b' & c' \\ -d' & e' & -f' \\ g' & -h' & i' \end{bmatrix}$$

TRANSPOSED COFACTOR MATRIX

$$\begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}$$

A MATRIX MULT. BY ITS TRANSP. COFACTOR IS ITS DET. SCALAR

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \frac{1}{D} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THE INVERSE OF A MATRIX IS ITS TRANSPOSED COFACTOR MATRIX DIVIDED BY ITS DETERMINANT

PROVIDED THAT ITS DETERMINANT IS NONZERO.

Say $D \neq 0$.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}$$

A MATRIX MULT. BY ITS TRANSP. COFACTOR IS ITS DET. SCALAR

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \frac{1}{D} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THE INVERSE OF A MATRIX IS ITS TRANSPOSED COFACTOR MATRIX DIVIDED BY ITS DETERMINANT

PROVIDED THAT ITS DETERMINANT IS NONZERO.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

FIND y ONLY

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \frac{1}{D} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} a & p & c \\ d & q & f \\ g & r & i \end{bmatrix} = -b'p + e'q - h'r$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{D} \begin{bmatrix} a' & -d' & g' \\ -b' & e' & -h' \\ c' & -f' & i' \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

STOP

CRAMER'S RULE

$$\frac{\det \begin{bmatrix} a & p & c \\ d & q & f \\ g & r & i \end{bmatrix}}{\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}} = \frac{-b'p + e'q - h'r}{D} = y$$

FIND ONLY