Financial Mathematics

Bilinear forms and quadratic forms

Definition:

A **polynomial in x**
is a (finite) linear combination of

$$\underline{1}, \underline{x}, \underline{x^2}, \underline{x^3}, x^4, x^5, x^6, x^7, \ldots$$

degree

e.g.: $$P(x) = 3 + 4x + 6x^2 - 2x^{\boxed{3}}.$$

Definition:

A **polynomial in x, y**
is a (finite) linear combination of

$$1, \quad x, y, \quad x^2, xy, y^2,$$

$$\boxed{x^3, x^2y, xy^2, y^3,} \quad \ldots$$

monomials in x, y of
(total) degree $= 3$

Definition:

A **polynomial in** x, y, z
is a (finite) linear combination of

$1,$ constant monomial (total degree 0)

$x, y, z,$ linear monomials (total degree 1)

$x^2, y^2, z^2, xy, xz, yz,$ quadratic monomials (total degree 2)

$\boxed{x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz},$

cubic monomials (total degree 3)

$\dots$

Definition:

A **polynomial in x, y, z**

is a (finite) linear combination of

$$1,$$

$$x, y, z,$$

$$x^2, y^2, z^2, xy, xz, yz,$$

$$\boxed{x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz},$$

$$\ldots$$

cubic monomials
(total degree 3)

Definition:

A **homogeneous polynomial in x, y, z of degree $= 3$**

is a linear combination of

$$x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz.$$

e.g.: $F(x,y,z) = 5x^3 + 3xz^2 - 4xyz$ $\mathbb{R}$-valued scalar-valued

three inputs one output

Definition: A function $G : \mathbb{R}^3 \to \mathbb{R}^{\boxed{2}}$ $\mathbb{R}^2$-valued vector-valued

is a **homogeneous polynomial of degree** $= 5$

if there are two homogeneous polynomials in $\boxed{x, y, z}$ of degree $= 5$, $A(\boxed{x, y, z})$ and $B(\boxed{x, y, z})$,

such that $G(\boxed{x, y, z}) = (\, A(\boxed{x, y, z})\, ,\, B(\boxed{x, y, z})\,)$.

Definition:

A **homogeneous polynomial in** x, y, z **of degree** $= 3$

is a linear combination of

$$x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz.$$

e.g.: $F(x, y, z) = 5x^3 + 3xz^2 - 4xyz$

three inputs one output

Definition: A function $G : \mathbb{R}^3 \to \mathbb{R}^2$ is a **homogeneous polynomial of degree** $= 5$

if there are two homogeneous polynomials in s, p, w of degree $= 5$, $A(s, p, w)$ and $B(s, p, w)$,

such that $G(s, p, w) = (\, A(s, p, w)\, ,\, B(s, p, w)\,)$.

---

etc., etc., etc.

Definition:

A **homogeneous polynomial in** x, y, z **of degree** $= 3$ is a linear combination of

$x^3, y^3, z^3, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz.$

Polynomial approximation

General mathematical theme:

Given a function, *e.g.*: $f(x) = e^x$,
even if it's not a polynomial,
we can approximate it by polynomials.

E.g.: $f(x) = e^x \approx 1$,

$$f(x) = e^x \approx 1 + x,$$

$$f(x) = e^x \approx 1 + x + (x^2/(2!)),$$

$$f(x) = \boxed{e^x} \approx \boxed{1 + x + (x^2/(2!)) + (x^3/(3!))},$$

$$\cdots$$

agree at 0 to order three

Works with any number of input variables,
and any number of output variables.

---

e.g.: Black-Scholes gives a function that maps
(spot, strike, risk-free rate, volatility)
four input
↦ (price, Delta)
two output

Denote this function by F.

S = spot
K = strike
ρ = risk-free rate
σ = volatility

$$K' := K/\rho$$

$$d_\pm := \frac{\ln(S/K')}{\sigma} \pm \frac{\sigma}{2}$$

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

---

price $= S[\Phi(d_+)] - K'[\Phi(d_-)]$

Delta $= \Phi(d_+)$

$F(S, K, \rho, \sigma) = ($ price , Delta $)$ highly nonlinear

---

e.g.: Black-Scholes gives a function that maps
(spot, strike, risk-free rate, volatility)
four input
$\underset{F}{\mapsto}$ (price, Delta)
two output

Denote this function by F.

Say we've computed $F(100, 97, 0.01, 0.2)$.

Say we want to compute

$$F(\ 100.1\ \ ,\ 97.03\ ,\ 0.0102\ ,\ 0.204\ \ ).$$

More generally, say we want to compute

$$F(100+\underline{w}, 97+\underline{x}, 0.01+\underline{y}, 0.2+\underline{z})$$

for "small" values of w, x, y, z.

---

e.g.: Black-Scholes gives a function that maps

(spot, strike, risk-free rate, volatility)

four input

$$\underset{F}{\longmapsto} \text{(price, Delta)}$$

two output

Denote this function by F.
Say we've computed $F(100, 97, 0.01, 0.2)$.

There are
a constant $C \in \mathbb{R}^2$,
a homogeneous linear $L : \mathbb{R}^4 \to \mathbb{R}^2$,
& a homogeneous quadratic $Q : \mathbb{R}^4 \to \mathbb{R}^2$,
s.t. $C + L(w,x,y,z) + Q(w,x,y,z)$
agrees, at $(0,0,0,0)$, to order two, with
$F(100+w, 97+x, 0.01+y, 0.2+z)$.

defined in topic
on multivariable
polynomial approximation

---

e.g.: Black-Scholes gives a function that maps
(spot, strike, risk-free rate, volatility)
four input
$\underset{F}{\longmapsto}$ (price, Delta)
two output

Denote this function by F.

Say we've computed $F(100, 97, 0.01, 0.2)$.

degree 0

degree 1

degree 2

There are

a constant $C \in \mathbb{R}^2$,

a homogeneous linear $L : \mathbb{R}^4 \to \mathbb{R}^2$,

& a homogeneous quadratic $Q : \mathbb{R}^4 \to \mathbb{R}^2$,

s.t. $C \; + \; L(w, x, y, z) \; + \; Q(w, x, y, z)$

agrees, at $(0, 0, 0, 0)$, to order two, with

$$G(w, x, y, z) := F(100 + w, 97 + x, 0.01 + y, 0.2 + z).$$

"2nd order Maclaurin approximation" of G

---

e.g.: Black-Scholes gives a function that maps

(spot, strike, risk-free rate, volatility)

four input

$\underset{F}{\mapsto}$ (price, Delta)

two output

degree 0

degree 1

degree 2

a constant $C \in \mathbb{R}^2$,

a homogeneous linear $L : \mathbb{R}^4 \to \mathbb{R}^2$,

& a homogeneous quadratic $Q : \mathbb{R}^4 \to \mathbb{R}^2$,

---

Constants are easy to understand.

Homogeneous linear functions are studied by "Linear Algebra".

degree 3

Do we need new subjects called "Quadratic Algebra", "Cubic Algebra", "Quartic Algebra", etc.?

degree 4

NO! There's a subject called
tensor algebra
which can be used to reduce
questions about degree ≥ 2 polynomials
to questions in linear algebra.
So linear algebra suffices!
It is the study of EVERYTHING!
We'll only need quadratic approximations,
where tensor algebra is particularly simple.

Constants are easy to understand.

Homogeneous linear functions are studied by
"Linear Algebra".

degree 3

Do we need new subjects called
"Quadratic Algebra", "Cubic Algebra",
"Quartic Algebra", etc.?

degree 4

Quadratic tensor algebra (naïve formulation)

vector-valued

Let $Q : \mathbb{R}^4 \to \boxed{\mathbb{R}^2}$ be homogeneous quadratic.

Then there are homogeneous quadratics

$$Q_1 : \mathbb{R}^4 \to \boxed{\mathbb{R}} \qquad \text{and} \qquad Q_2 : \mathbb{R}^4 \to \boxed{\mathbb{R}}$$

such that scalar-valued scalar-valued

$$Q(w,x,y,z) = \big(Q_1(w,x,y,z) \;,\; Q_2(w,x,y,z) \big).$$

It's enough to understand quadratic functions $\mathbb{R}^n \to \mathbb{R}$

Definition: A homogeneous quadratic function $\mathbb{R}^n \to \mathbb{R}$ is called a **quadratic form**.

e.g.: $F(s,t,u,v,w,x) = 4st - 2uv + u^2 - 3v^2 + wx$

e.g.: $F(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$

Also denoted: $v \mapsto |v|^2 : \mathbb{R}^n \to \mathbb{R}$

Bilinear Forms

Def'n: A fn $B : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is a **bilinear form** if both of the following two conditions hold:

- for all $v \in \mathbb{R}^p$, the function $B(v, \bullet) : \mathbb{R}^q \to \mathbb{R}$ is linear; and,
- for all $w \in \mathbb{R}^q$, the function $B(\bullet, w) : \mathbb{R}^p \to \mathbb{R}$ is linear.

Let $e_1, \ldots, e_p$ be the standard basis of $\mathbb{R}^p$.

$$e_1 := \underset{\underset{\mathbb{R}^p}{\cap}}{(1, 0, \ldots, 0)}, \quad \ldots, \quad e_p := \underset{\underset{\mathbb{R}^p}{\cap}}{(0, \ldots, 0, 1)}$$

Let $f_1, \ldots, f_q$ be the standard basis of $\mathbb{R}^q$.

$$f_1 := \underset{\underset{\mathbb{R}^q}{\cap}}{(1, 0, \ldots, 0)}, \quad \ldots, \quad f_q := \underset{\underset{\mathbb{R}^q}{\cap}}{(0, \ldots, 0, 1)}$$

Bilinear Forms

a bilinear form

Let $B : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}$ be bilinear.
Let $e_1, \ldots e_p$ be the standard basis of $\mathbb{R}^p$.
Let $f_1, \ldots f_q$ be the standard basis of $\mathbb{R}^q$.

The **matrix of** B is
the $p \times q$ matrix whose (j, k) entry is
$B(e_j, f_k)$.

The matrix of B is denoted $\boxed{[B]}$.

$$[B] := [B(e_j, f_k)]_{j=1,\ldots,p}^{k=1,\ldots,q}.$$

Bilinear Forms

a bilinear form

Let $B : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be bilinear.

Let $e_1, \ldots e_p$ be the standard basis of $\mathbb{R}^p$.

Let $f_1, \ldots f_q$ be the standard basis of $\mathbb{R}^q$.

The **matrix of** B is

$$\boxed{[B]} := \begin{bmatrix} B(e_1, f_1) & B(e_1, f_2) & \cdots & B(e_1, f_q) \\ B(e_2, f_1) & B(e_2, f_2) & \cdots & B(e_2, f_q) \\ \vdots & \vdots & & \vdots \\ B(e_p, f_1) & B(e_p, f_2) & \cdots & B(e_p, f_q) \end{bmatrix}$$

Next:
A bilinear form is determined by its matrix

A bilinear form is determined by its matrix

Let $B : \mathbb{R}^3 \times \mathbb{R}^5 \longrightarrow \mathbb{R}$ be bilinear.

A bilinear form is determined by its matrix

A bilinear form is determined by its matrix

Let $B : \mathbb{R}^3 \times \mathbb{R}^5 \to \mathbb{R}$ be bilinear.

Suppose the matrix of B is

$$[B] = \begin{array}{cc} \begin{array}{ccccc} 3 & 7 & -5 & 6 & -2 \end{array} & \\ \left[\begin{array}{ccccc} \boxed{2} & \boxed{5} & 3 & -9 & 1 \\ \boxed{-5} & 4 & 2 & 1 & 0 \\ 7 & -6 & 4 & -5 & 2 \end{array}\right] & \begin{array}{c} 2 \\ 2 \\ 1 \end{array} \end{array}$$

Problem: Compute

$$B\left(\quad(2,2,1)\quad,\quad(3,7,-5,6,-2)\quad\right)$$

$$\|$$

$$(2)(3)(2) + (5)(7)(2) + (3)(-5)(2) + (-9)(6)(2) + (1)(-2)(2)$$
$$+(-5)(3)(2) + (4)(7)(2) + (2)(-5)(2) + (1)(6)(2) + (0)(-2)(2)$$
$$+(7)(3)(1) + (-6)(7)(1) + (4)(-5)(1) + (-5)(6)(1) + (2)(-2)(1)$$

Exercise: Show $B(v,w) = (L_{[B]}(w)) \cdot v.$

Symmetric Bilinear Forms

equal

Let $B : \mathbb{R}^{\boxed{n}} \times \mathbb{R}^{\boxed{n}} \to \mathbb{R}$ be bilinear.

Def'n: We say B is **symmetric** if,
for all $v, v' \in \mathbb{R}^n$,

$$B(v, v') = B(v', v).$$

---

e.g.: The dot product

$$(v, v') \mapsto v \cdot v' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}.$$

---

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

symmetric matrix

Fact: A bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric iff its matrix is symmetric.

First study how to go from $[B]$ to Q

Symmetric Bilinear Forms

Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear.

Def'n: We say B is **symmetric** if,
for all $v, v' \in \mathbb{R}^n$,

$$B(v, v') = B(v', v).$$

Note: If $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form,
then the function $Q : \mathbb{R}^n \to \mathbb{R}$

defined by $Q(v) = B(\boxed{v, v})$

on the "diagonal"

is a quadratic form.

Question:

Can we go from Q to B, (*i.e.*, to $[B]$),
thereby reducing questions about Q
to matrix questions?

First study how to go from $[B]$ to Q Different $[B]$ lead to the same Q

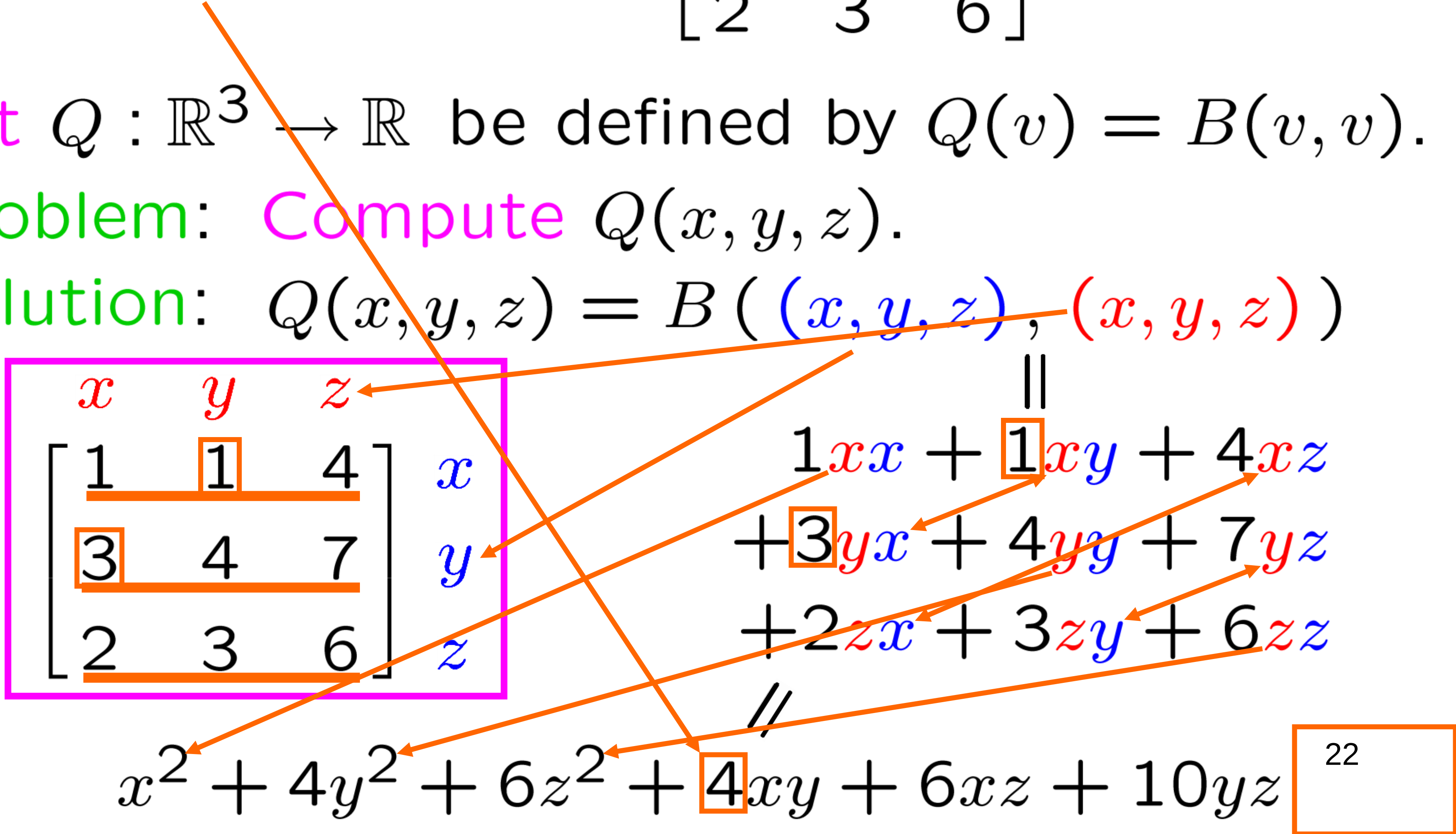
e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & \boxed{1} & 4 \\ \boxed{3} & 4 & 7 \\ 2 & 3 & 6 \end{bmatrix}$$

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v, v)$.

Problem: Compute $Q(x, y, z)$.

Solution: $Q(x, y, z) = B\left((x, y, z), (x, y, z) \right)$



$$\begin{array}{cccc} x & y & z & \\ 1 & \boxed{1} & 4 & x \\ \boxed{3} & 4 & 7 & y \\ 2 & 3 & 6 & z \end{array}$$

$$\| $$

$$1xx + \boxed{1}xy + 4xz + \boxed{3}yx + 4yy + 7yz + 2zx + 3zy + 6zz$$

$$/\!/$$

$$x^2 + 4y^2 + 6z^2 + \boxed{4}xy + 6xz + 10yz$$

First study how to go from $[B]$ to Q Different $[B]$ lead to the same Q

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & \boxed{2} & \boxed{4} \\ \boxed{2} & 4 & 7 \\ \boxed{2} & 3 & 6 \end{bmatrix}$$

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v, v)$.

Problem: Compute $Q(x, y, z)$.

Solution: $Q(x, y, z) = B\,(\,(x, y, z)\,,\,(x, y, z)\,)$

$$\boxed{\begin{array}{cc} \begin{matrix} x & \quad y & \quad z \end{matrix} & \\ \begin{bmatrix} 1 & \boxed{2} & \boxed{4} \\ \boxed{2} & 4 & 7 \\ \boxed{2} & 3 & 6 \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}}$$

$$\begin{aligned} &= 1xx + \boxed{2}xy + \boxed{4}xz \\ &\quad + \boxed{2}yx + 4yy + 7yz \\ &\quad + \boxed{2}zx + 3zy + 6zz \end{aligned}$$

$$= x^2 + 4y^2 + 6z^2 + \boxed{4}xy + \boxed{6}xz + 10yz$$

First study how to go from $[B]$ to Q Different $[B]$ lead to the same Q

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & 2 & \boxed{3} \\ 2 & 4 & \boxed{7} \\ \boxed{3} & \boxed{3} & 6 \end{bmatrix}$$

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v,v)$.

Problem: Compute $Q(x,y,z)$.

Solution: $Q(x,y,z) = B\big(\,(x,y,z)\,,\,(x,y,z)\,\big)$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} & \\ \begin{bmatrix} 1 & 2 & \boxed{3} \\ 2 & 4 & \boxed{7} \\ \boxed{3} & \boxed{3} & 6 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

$$\|$$

$$1xx + 2xy + \boxed{3}xz$$
$$+2yx + 4yy + \boxed{7}yz$$
$$+\boxed{3}zx + \boxed{3}zy + 6zz$$

$$/\!/$$

$$x^2 + 4y^2 + 6z^2 + 4xy + \boxed{6}xz + \boxed{10}yz$$

First study how to go from $[B]$ to Q Different $[B]$ lead to the same Q

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & \boxed{5} \\ 3 & \boxed{5} & 6 \end{bmatrix}$$

symmetric

Coming up: The Spectral Theorem

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v, v)$.

Problem: Compute $Q(x, y, z)$.

Solution: $Q(x, y, z) = B\left((x, y, z), (x, y, z) \right)$

$$\boxed{\begin{array}{cc} \begin{matrix} x & y & z \end{matrix} & \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & \boxed{5} \\ 3 & \boxed{5} & 6 \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}}$$

$$\|$$

$$1xx + 2xy + 3xz$$
$$+2yx + 4yy + \boxed{5}yz$$
$$+3zx + \boxed{5}zy + 6zz$$

$$//$$

$$x^2 + 4y^2 + 6z^2 + 4xy + 6xz + \boxed{10}yz$$

First study how to go from $[B]$ to Q Different $[B]$ lead to the same Q

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v, v)$.

$$\begin{tikzcd} & \{\text{square matrices}\} \arrow[dl, "\text{not one-to-one!}"'] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\ \{\text{quadratic forms}\} & \{\text{symmetric matrices}\} \end{tikzcd}$$

FDifferent $[B]$ lead to the same Q ⊃ Q Different $[B]$ lead to the same Qn Q to $[B]$

e.g.: Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the bilinear form whose matrix is

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be defined by $Q(v) = B(v, v)$.

$$\begin{tikzcd}
 & \{\text{square matrices}\} \arrow[dl, "\text{not one-to-one!}"'] \\
\{\text{quadratic forms}\} & \{\text{symmetric matrices}\} \arrow[l, "\text{bijection}"] \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Different $[B]$ lead to the same Q

Now learn to go from Q to $[B]$ and B

Symmetric Bilinear Forms

symmetry is needed!

Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear.

We say B is **symmetric** if,
for all $v, v' \in \mathbb{R}^n$,

$$\underline{B(v,v') = B(v',v)}.$$

$$\begin{tikzcd}
 & \{\text{square matrices}\} \arrow[dl, "\text{not one-to-one!}"'] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
\{\text{quadratic forms}\} \arrow[r, shift left, "?"] & \{\text{symmetric matrices}\} \arrow[l, shift left, "\text{bijection}"]
\end{tikzcd}$$

Different $[B]$ lead to the same Q

Now learn to go from Q to $[B]$ and B

Symmetric Bilinear Forms

symmetry is needed!

Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear.

We say B is **symmetric** if,
for all $v, v' \in \mathbb{R}^n$,

$$B(v, v') = B(v', v).$$

From B to Q

Note: If $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form,
then the function $Q : \mathbb{R}^n \to \mathbb{R}$
defined by $Q(v) = B(v, v)$
is a quadratic form.

From Q to B

Goal: If $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form,
then $\exists\ \boxed{!}\ \boxed{\text{SBF}}\ B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
s.t. $\forall v \in \mathbb{R}^n$,

$$Q(v) = B(v, v).$$

unique

symmetric bilinear form

Now learn to go from Q to $[B]$
and B

symmetry
is needed!

e.g.: Define $Q : \mathbb{R}^4 \to \mathbb{R}$ by
$$Q(w,x,y,z) = \boxed{w^2}\boxed{+\,6wx}\boxed{-\,4wz}\boxed{+\,3y^2}\boxed{-\,2yz}\boxed{+\,7z^2}.$$

$$[B] = \begin{array}{ccccc} w & x & y & z & \\ 1 & 3 & ? & -2 & w \\ 3 & 0 & 0 & 0 & x \\ ? & 0 & 3 & -1 & y \\ -2 & 0 & -1 & 7 & z \end{array}$$

Goal: If $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form,
then $\exists\ \boxed{!}\ \boxed{\text{SBF}}\ B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
s.t. $\forall v \in \mathbb{R}^n$,
$$Q(v) = B(v,v).$$

unique

symmetric bilinear form

Now recall how to go from $[B]$ to B

Now learn to go from Q to $[B]$ $\boxed{\text{and } B}$

symmetry
is needed!

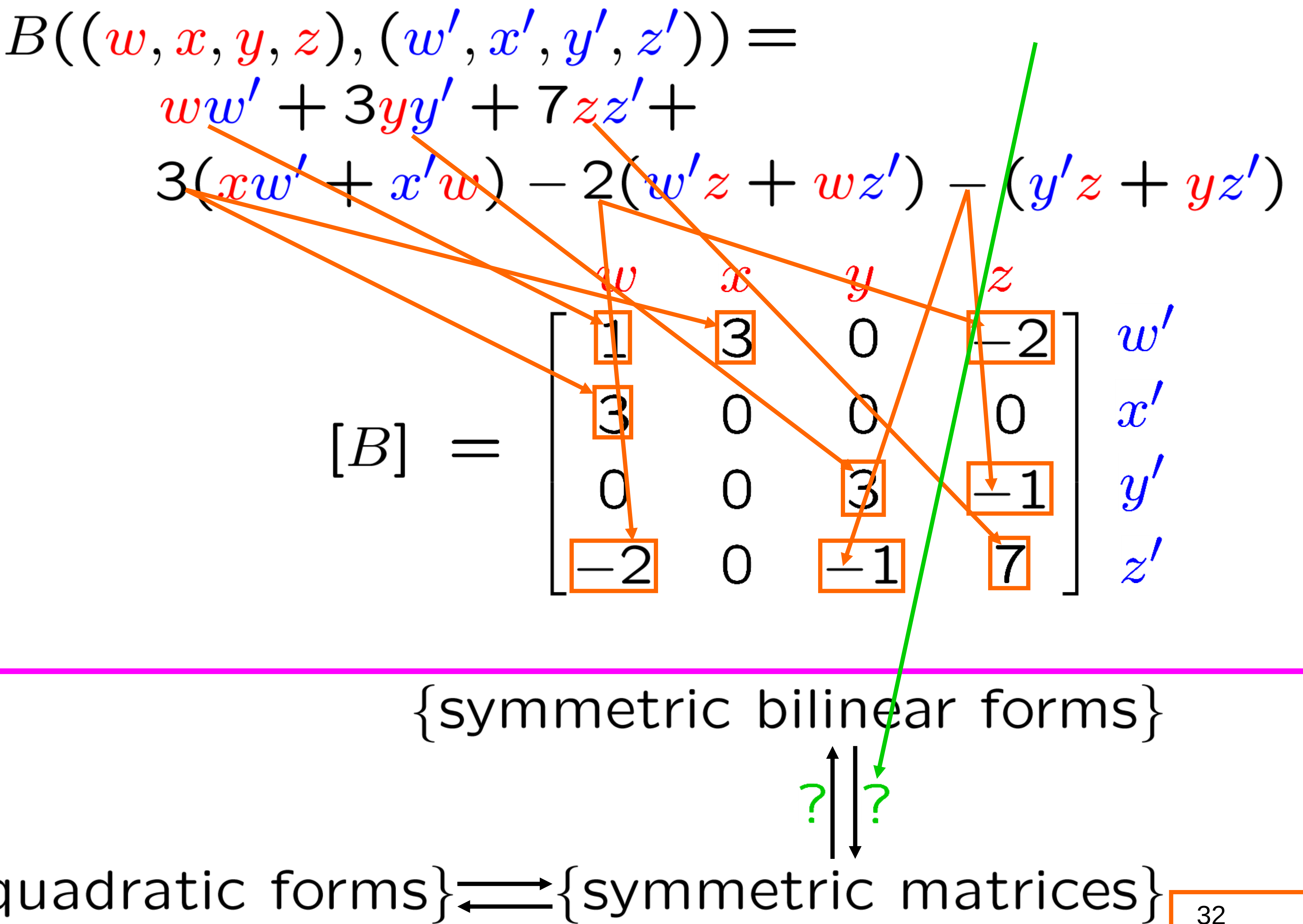
e.g.: Define $Q : \mathbb{R}^4 \to \mathbb{R}$ by

$$Q(w,x,y,z) = w^2 + 6wx - 4wz + 3y^2 - 2yz + 7z^2.$$

$$[B] = \begin{array}{cc} \begin{matrix} \;w\; & \;x\; & \;y\; & \;z\; \end{matrix} & \\ \begin{bmatrix} 1 & 3 & 0 & -2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ -2 & 0 & -1 & 7 \end{bmatrix} & \begin{matrix} w \\ x \\ y \\ z \end{matrix} \end{array}$$

{quadratic forms} $\rightleftarrows$ {symmetric matrices}

Now recall how to go from $[B]$ to B Now learn how to go from B to $[B]$



$$B((w,x,y,z),(w',x',y',z')) =$$
$$ww' + 3yy' + 7zz' +$$
$$3(xw' + x'w) - 2(w'z + wz') - (y'z + yz')$$

$$[B] \;=\; \begin{array}{cccc|c} w & x & y & z & \\ \boxed{1} & \boxed{3} & 0 & \boxed{-2} & w' \\ \boxed{3} & 0 & 0 & 0 & x' \\ 0 & 0 & \boxed{3} & \boxed{-1} & y' \\ \boxed{-2} & 0 & \boxed{-1} & \boxed{7} & z' \end{array}$$

$$\begin{tikzcd} & \{\text{symmetric bilinear forms}\} \arrow[d, shift left, "?"] \\ \{\text{quadratic forms}\} \arrow[r, shift left] & \{\text{symmetric matrices}\} \arrow[l, shift left] \arrow[u, shift left, "?"] \end{tikzcd}$$

Now recall how to go from B to Q Now learn how to go from B to $[B]$

$$B((w,x,y,z),(w',x',y',z')) = \\ ww' + 3yy' + 7zz' + \\ 3(xw' + x'w) - 2(w'z + wz') - (y'z + yz')$$

$$[B] = \begin{array}{cc} \begin{array}{cccc} w & x & y & z \end{array} & \\ \begin{bmatrix} 1 & 3 & 0 & -2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ -2 & 0 & -1 & 7 \end{bmatrix} & \begin{array}{c} w' \\ x' \\ y' \\ z' \end{array} \end{array}$$

$$\begin{tikzcd} & \{\text{symmetric bilinear forms}\} \arrow[d, shift left] \\ \{\text{quadratic forms}\} \arrow[r, shift left] & \{\text{symmetric matrices}\} \arrow[u, shift left] \arrow[l, shift left] \end{tikzcd}$$

Now recall how to go from B to Q Last: Q to B

Note: If $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form,
then the function $Q : \mathbb{R}^n \to \mathbb{R}$
defined by $\underline{Q(v) = B(v, v)}$
is a quadratic form.

$B(v, w) =$ Formula involving Q

$v :\to v + w$

$$Q(\boxed{v}) = B(\boxed{v}, \boxed{v}) \qquad Q(\boxed{w}) = B(\boxed{w}, \boxed{w})$$

$$Q(\boxed{v + w}) = B(\boxed{\boxed{v} + \boxed{w}}, \boxed{\boxed{v} + \boxed{w}})$$

← expand to four terms

symmetry

$$= \underline{B(\boxed{v}, \boxed{v})} + \underline{B(\boxed{v}, \boxed{w}) + B(\boxed{w}, \boxed{v})} + \underline{B(\boxed{w}, \boxed{w})}$$

$$= \underline{Q(v)} + \underline{2[B(v, w)]} + \underline{Q(w)}$$

$$\begin{tikzcd}
 & \{\text{symmetric bilinear forms}\} \arrow[dl, "\text{restr. to diagonal}"'] \arrow[d, shift left] \\
\{\text{quadratic forms}\} \arrow[ur, shift right, "\text{Formula?}"'] \arrow[r, shift left] & \{\text{symmetric matrices}\} \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

Last: Q to B ☺

Note: If $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form, then the function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(v) = B(v,v)$ is a quadratic form.

$$B(v,w) = \frac{[Q(v+w)] - [Q(v)] - [Q(w)]}{2}$$

Formula involving Q

$$[Q(v+w)] - [Q(v)] - [Q(w)]$$

$$= Q(v) + 2[B(v,w)] + Q(w)$$

DIVIDE BY 2

$$\begin{tikzcd}
& \{\text{symmetric bilinear forms}\} \arrow[dl, shift right, "\text{restr. to diagonal}"'] \arrow[d, shift left] \\
\{\text{quadratic forms}\} \arrow[ur, shift right, "\text{Formula?}"'] \arrow[r, shift left] & \{\text{symmetric matrices}\} \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

Def'n: The SBF B s.t., $\forall v$, $Q(v) = B(v,v)$ is called the **polarization of** Q.

The Polarization Formula

$$B(v,w) = \frac{[Q(v+w)] - [Q(v)] - [Q(w)]}{2}$$

$$Q(v+w) \neq Q(v) \qquad + \qquad Q(w)$$

$$Q(v+w) = Q(v) + 2[\boxed{B(v,w)}] + Q(w)$$

the "cross term"

$$\begin{tikzcd}
 & \{\text{symmetric bilinear forms}\} \arrow[dl, shift right, "\text{restr. to diagonal}"'] \arrow[d, shift left] \\
\{\text{quadratic forms}\} \arrow[ur, shift right, "\text{polarization}"'] \arrow[r, shift left] & \{\text{symmetric matrices}\} \arrow[u, shift left] \arrow[l, shift left]
\end{tikzcd}$$

Def'n: The SBF B s.t., $\forall v$, $Q(v) = B(v,v)$ is called the **polarization of** Q.

The Polarization Formula

$$B(v,w) = \frac{[Q(v+w)] - [Q(v)] - [Q(w)]}{2}$$

Def'n: For any quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, with polarization B, the **matrix of** Q, denoted $\boxed{[Q]}$, is the matrix of B.

$B_M(v,w) = (L_M(v)) \cdot w$

$Q_M(v) = (L_M(v)) \cdot v$

$$\begin{tikzcd} & \{\text{symmetric bilinear forms}\} \arrow[d, shift left, "{B \mapsto [B]}"] \\ \{\text{quadratic forms}\} \arrow[r, shift left, "{Q \mapsto [Q]}"] & \{\text{symmetric matrices}\} \arrow[l, shift left, "{Q_M \leftarrow\!\shortmid M}"] \arrow[u, shift left, "{B_M \leftarrow\!\shortmid M}"] \end{tikzcd}$$

Polarization of squaring is multiplication

e.g.:
Define $Q:\mathbb{R}\to\mathbb{R}$ by $Q(x)=x^2$.

The polarization formula:

$$B(x,y)=\frac{[Q(x+y)]-[Q(x)]-[Q(y)]}{2}$$

$$=\frac{(x+y)^2-x^2-y^2}{2}$$

$$=\frac{(\cancel{x^2}+\cancel{y^2}+\cancel{2}xy)\cancel{-x^2}\cancel{-y^2}}{\cancel{2}}=xy$$

On your calculator,
if the × button breaks, but not the x^2 button,
you can still multiply!

Polarization of length squared is dot product

e.g.:

Define $Q : \mathbb{R}^n \to \mathbb{R}$ by $Q(v) = |v|^2$.

The polarization formula:

$$B(v,w) = \frac{[Q(v+w)] - [Q(v)] - [Q(w)]}{2}$$

$$= \frac{|v+w|^2 - |v|^2 - |w|^2}{2}$$

$$= \frac{(\cancel{|v|^2} + \cancel{|w|^2} + \cancel{2}v \cdot w) \cancel{- |v|^2} \cancel{- |w|^2}}{\cancel{2}} = v \cdot w$$

If you know how to compute $| \bullet |^2$,

then you know how to compute dot products.

The BIG IDEA:

Questions about quadratic forms reduce to
questions about (symmetric) bilinear forms,
which then reduce to
questions about (symmetric) matrices.

Thus "quadratic algebra" reduces to
linear algebra!!

Similarly:

Questions about $\overbrace{\text{cubic forms}}^{\boxed{\begin{array}{c}\text{homogeneous polynomials}\\ \text{of degree} = 3\end{array}}}$ reduce to
questions about (symmetric) trilinear forms,
which then reduce to
questions about (symmetric) 3-tensors.

Thus "cubic algebra" reduces to
linear algebra!!

The BIG IDEA:

Questions about homogeneous polynomials of degree k reduce to questions about (symmetric) k-linear forms, which then reduce to questions about (symmetric) k-tensors.

Thus "polynomial algebra" reduces to linear algebra!!

Similarly:

Questions about cubic forms reduce to questions about (symmetric) trilinear forms, which then reduce to questions about (symmetric) 3-tensors.

Thus "cubic algebra" reduces to linear algebra!!

The BIG IDEA:

Questions about homogeneous polynomials of degree k reduce to questions about (symmetric) k-linear forms, which then reduce to questions about (symmetric) k-tensors.

Thus "polynomial algebra" reduces to linear algebra!!

STOP

Concluding remark:

Since any reasonable function is well-approximated by polynomials, these observations turn linear algebra into the study of everything!!