Financial Mathematics

Cauchy-Schwarz

Cauchy-Schwarz inequality

Can $v \cdot w$ be big, even if $|v|$ and $|w|$ are small?

Say we know $|v| \leq 3$ and $|w| \leq 5$.
How big can $v \cdot w$ be?

Say we know $|v|^2 \leq 9$ and $|w|^2 \leq 25$.
How big can $v \cdot w$ be?

Cauchy-Schwarz inequality

$$|v|^2 \leq 9$$
$$|w|^2 \leq 25$$

$$0 \leq |v - w|^2$$

$$|v|^2 \leq 9 \qquad |w|^2 \leq 25$$

Cauchy-Schwarz inequality

DIVIDE BY 2

$|v|^2 \le 9$

$|w|^2 \le 25$

$0 \le |v-w|^2 = |v|^2 + |w|^2 - 2v \cdot w$

$|\bullet|^2$ is positive semidefinite

$v :\to \sqrt{t}v$

$w :\to w/\sqrt{t}$

$$v \cdot w \le \frac{|v|^2 + |w|^2}{2} \le \frac{9+25}{2} = 17$$

$$v \cdot w = 3v \cdot \frac{w}{3} \le \frac{9|v|^2 + (|w|^2/9)}{2} \le \frac{377}{9}$$

$$v \cdot w = \sqrt{5}v \cdot \frac{w}{\sqrt{5}} \le \frac{5|v|^2 + (|w|^2/5)}{2} \le 25$$

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \le \frac{t|v|^2 + (|w|^2/t)}{2} \le \frac{9t + (25/t)}{2}$$

minimum value?

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2}$$

$$|v|^2 \leq 9$$

$$|w|^2 \leq 25$$

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$$\frac{1}{2}a + \frac{1}{2}b$$

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2}$$

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \boxed{\frac{t|v|^2 + (|w|^2/t)}{2}}$$

$$\boxed{\begin{matrix} |v|^2 \leq 9 \\ |w|^2 \leq 25 \end{matrix}}$$

$$v \cdot w \leq \sqrt{|v|^2|w|^2} = |v||w|$$

$$\frac{1}{2}a + \frac{1}{2}b$$

$\boxed{ta}$ $\boxed{b/t}$

$a :\rightarrow |v|^2$

$b :\rightarrow |w|^2$

Fact: $\forall a, b \in (0, \infty], \quad \min_{t>0} \boxed{\left[\frac{t}{2}a + \frac{1}{2t}b\right]} = \sqrt{ab}$

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2}$$

$|v|^2 \leq 9$

$|w|^2 \leq 25$

$v :\to -v$

$$v \cdot w \leq \sqrt{|v|^2|w|^2} = |v||w|$$

$|v| \leq 3$

$$v \cdot w \leq |v||w| \leq (3)(5) = 15$$

$|w| \leq 5$

$$-(v \cdot w) = (-v) \cdot w \leq |-v||w| = |v||w|$$

$$|v \cdot w| = \max\{v \cdot w, -(v \cdot w)\} \leq |v||w|$$

$$|x| = \max\{\ x\ ,\ -x\ \}$$

Cauchy-Schwarz inequality:

$$|v \cdot w| \leq |v||w|$$

Cauchy-Schwarz inequality:

$$\boxed{|v \cdot w| \leq |v||w|}$$

$$|v \cdot w| \leq \sqrt{|v|^2}\sqrt{|w|^2}$$

Cauchy-Schwarz inequality:

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Cauchy-Schwarz inequality:

$$|v \cdot w| \leq |v||w|$$

$$|v \cdot w| \leq \sqrt{|v|^2}\sqrt{|w|^2}$$

Q is positive semidefinite

Theorem:

Let $Q : V \to \mathbb{R}$ be a quadratic form.

Assume, for all $v \in V$, that $Q(v) \geq 0$.

Let B be the polarization of Q.

Then $|B(v, w)| \underset{?}{\leq} \sqrt{Q(v)}\sqrt{Q(w)}$.

What have I forgotten?

Def'n: A quadratic form $Q : V \to \mathbb{R}$ is **positive semidefinite** if, $\forall v \in V$, $Q(v) \geq 0$.

Def'n: A quadratic form $Q : V \to \mathbb{R}$ is **positive definite** if, $\forall v \in V \backslash \{0\}$, $Q(v) > 0$.

Theorem:

Let $Q : V \longrightarrow \mathbb{R}$ be a quadratic form.

Assume that Q is positive semidefinite.

Let B be the polarization of Q.

Then $\boxed{|B(v, w)| \leq \sqrt{Q(v)}\sqrt{Q(w)}}$.

Cauchy-Schwarz inequality for B and Q

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Def'n: A quadratic form $Q : V \to \mathbb{R}$ is **positive definite** if, $\forall v \in V \backslash \{0\}$, $Q(v) > 0$.

Theorem:

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the geometric mean of $Q(v)$ and $Q(w)$

The absolute polarization at v, w of a $\overbrace{\text{Q. form}}^{\text{positive semidefinite}}$

$\leq$

the geometric mean of its values at v and w

STOP

Theorem:

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Assume that Q is positive semidefinite.

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Then $|B(v, w)| \leq \boxed{\sqrt{Q(v)}\sqrt{Q(w)}}$.

the geometric mean of $Q(v)$ and $Q(w)$