Financial Mathematics

Rotations, reflections and orthogonal transformations

Definition:
The **length of a vector** $v = (a_1, \ldots, a_k)$ is

$$\boxed{|v|} := \sqrt{a_1^2 + \cdots + a_k^2}$$

Note: $v \cdot v = a_1^2 + \cdots + a_k^2 = |v|^2$

Definition: v is **normal** if $|v| = 1$.

Definition: v and w are **orthogonal**, written $\boxed{v \perp w}$, if $v \cdot w = 0$.

Definition:
A collection $v_1, \ldots, v_m$ of vectors is **orthonormal** if both of the following hold:

- for all integers $i \in [1, n]$, $|v_i| = 1$
- for all integers $i, j \in [1, n]$, $i \neq j$ implies $v_i \perp v_j$.

Definition:

The **Kronecker delta** is defined by

$$\boxed{\delta_j^k} = \begin{cases} 0, \text{ if } j \neq k \\ 1, \text{ if } j = k. \end{cases}$$

Definition:

A collection of vectors $v_1, \ldots, v_n$ is **orthonormal** if, $\forall$integers $j, k \in [1, n]$,

$$v_j \cdot v_k = \delta_j^k.$$

i.e.: $j \neq k$ implies $v_j \perp v_k$ (pairwise) orthogonal

and

$\forall j$, $|v_j| = 1$. normal

Fact: A (real) square matrix M is orthogonal
if and only if
the columns of M "are" orthonormal.

Proof: M is orthogonal iff $M^tM = I$.

$\forall$integers $j \in [1, n]$,
let $v_j \in \mathbb{R}^n$ be the vector
whose entries are the entries of
the jth column of M.

only if: $\forall$integers $j, k \in [1, n]$,
the jth row of M^t "is" v_j and
the kth column of M "is" v_k,
so the (j, k) entry of M^tM is δ_j^k.

if: Similar. QED

M is orthogonal

Fact: A (real) square matrix M is orthogonal

iff $M^tM = I$.

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if and only if

M is orthogonal iff $M^tM = I$.

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Fact: Let $X, Y \in \mathbb{R}^{n\times n}$. Assume that $XY = I$.

Then $YX = I$.

M is orthogonal

iff $M^t M = I$ iff $M M^t = I$.

Fact: A (real) square matrix M is orthogonal
if and only if
the columns of M "are" orthonormal.

Fact: Let $X, Y \in \mathbb{R}^{n \times n}$. Assume that $XY = I$.
Then $YX = I$.

M is orthogonal

iff $M M^t = I$.

Fact: A (real) square matrix M is orthogonal
if and only if
the rows of M "are" orthonormal.

Gram-Schmidt Orthonormalization

Definition: The **flag of** $v_1, \ldots, v_n$ is

$$\langle v_1 \rangle, \quad \langle v_1, v_2 \rangle, \quad \langle v_1, v_2, v_3 \rangle, \quad \ldots, \quad \langle v_1, \ldots, v_n \rangle.$$

Gram-Schmidt attempts to replace $v_1, \ldots, v_n$ with an o.n. $w_1, \ldots, w_n$ with the same flag.

orthonormal

$x_1 := v_1$

$w_1 := x_1/|x_1|$ normal

$x_2 := v_2 - (w_1 \cdot v_2)w_1$ $\perp w_1$

$w_2 := x_2/|x_2|$ normal and orthog. to w_1

$x_3 := v_3 - (w_1 \cdot v_3)w_1 - (w_2 \cdot v_3)w_2$ $\perp w_1, w_2$

$w_3 := x_3/|x_3|$ normal and orthog. to w_1, w_2

etc.

WARNING: If $x_k = 0$, then STOP.

e.g.: $v_1 = (1,2,0,-1)$, $v_2 = (2,3,-1,1)$, $v_3 = (1,1,2,4)$

$|v_1| = \sqrt{6}$ $\quad \langle w_1 \rangle = \langle v_1 \rangle$

$a := 1/\sqrt{6}$ $\quad w_1 := a(1,2,0,-1)$ normal

$$(2,3,-1,1) - \underbrace{a(2+6+0-1)}_{w_1 \cdot v_2}\underbrace{a(1,2,0,-1)}_{w_1} =$$

$\perp w_1$

$$(2,3,-1,1) - 7a^2(1,2,0,-1) =$$

$$(2-(7/6), 3-(14/6), -1, 1+(7/6)) =$$

same as normalizing $(5,4,-6,13)$

normalize: $(5,4,-6,13)/6 \perp w_1$

$$b := 1/\sqrt{5^2+4^2+6^2+13^2} = 1/\sqrt{246}$$

$\langle w_1, w_2 \rangle = \langle v_1, v_2 \rangle$ $\quad w_2 := b(5,4,-6,13)$ normal $\perp w_1$

$$(1,1,2,4) - \underbrace{a(1+2+0-4)}_{w_1 \cdot v_3}\underbrace{a(1,2,0,-1)}_{w_1} -$$

$$-\underbrace{b(5+4-12+52)}_{w_2 \cdot v_3}\underbrace{b(5,4,-6,13)}_{w_2} =$$

$\perp w_1, w_2$

$$(1,1,2,4) + a^2(1,2,0,-1) - 49b^2(5,4,-6,13)$$

$a^2 = 1/6$ $= 41/246$ $\quad 49b^2 = 49/246$

e.g.: $v_1 = (1, 2, 0, -1)$, $v_2 = (2, 3, -1, 1)$, $v_3 = (1, 1, 2, 4)$

$$a := 1/\sqrt{6} \qquad w_1 := a(1, 2, 0, -1)$$

$$b := 1/\sqrt{246} \qquad w_2 := b(5, 4, -6, 13)$$

$$(1, 1, 2, 4) - a(1 + 2 + 0 - 4)a(1, 2, 0, -1) - $$
$$-b(5 + 4 - 12 + 52)b(5, 4, -6, 13) =$$

$$(1, 1, 2, 4) + a^2(1, 2, 0, -1) - 49b^2(5, 4, -6, 13)$$

$b :=$ $a^2 = 1/6$ $49b^2 = 49/246$

$= 41/246$ $w_2 := b(5, 4, -6, 13)$

$$\frac{246(1, 1, 2, 4) + 41(1, 2, 0, -1) - 49(5, 4, -6, 13)}{246} =$$

$$-b(5 + 4 - 12 + 52)b(5, 4, -6, 13) =$$

$$(1, 1, 2, 4) + a^2(1, 2, 0, -1) - 49b^2(5, 4, -6, 13)$$

$a^2 = 1/6$ $49b^2 = 49/246$

$= 41/246$

e.g.: $v_1 = (1, 2, 0, -1)$, $v_2 = (2, 3, -1, 1)$, $v_3 = (1, 1, 2, 4)$

$$a := 1/\sqrt{6} \qquad w_1 := a(1, 2, 0, -1)$$

$$b := 1/\sqrt{246} \qquad w_2 := b(5, 4, -6, 13)$$

$$(1, 1, 2, 4) - a(1 + 2 + 0 - 4)a(1, 2, 0, -1) - \\ -b(5 + 4 - 12 + 52)b(5, 4, -6, 13) =$$

$$(1, 1, 2, 4) + a^2(1, 2, 0, -1) - 49b^2(5, 4, -6, 13)$$

$a^2 = 1/6 = 41/246$ $\qquad$ $49b^2 = 49/246$

$$\frac{246(1, 1, 2, 4) + 41(1, 2, 0, -1) - 49(5, 4, -6, 13)}{246} =$$

$$\frac{(42, 132, 786, 306)}{246} = \boxed{\frac{(7, 22, 131, 51)}{41}}$$

Normalize

Normalize: $(7, 22, 131, 51)$

e.g.: $v_1 = (1,2,0,-1)$, $v_2 = (2,3,-1,1)$, $v_3 = (1,1,2,4)$

$a := 1/\sqrt{6}$ $\qquad$ $w_1 := a(1,2,0,-1)$

$b := 1/\sqrt{246}$ $\qquad$ $w_2 := b(5,4,-6,13)$

$$(1,1,2,4) - a(1+2+0-4)a(1,2,0,-1) - \\ -b(5+4-12+52)b(5,4,-6,13) =$$

$$(1,1,2,4) + a^2(1,2,0,-1) - 49b^2(5,4,-6,13)$$

$a^2 = 1/6$
$= 41/246$

$49b^2 = 49/246$

$$c := 1/\sqrt{7^2 + 22^2 + 131^2 + 51^2} \\ = 1/\sqrt{20,295}$$

$$w_3 := c(7,22,131,51)$$

Normalize: $(7,22,131,51)$

e.g.: $v_1 = (1, 2, 0, -1)$, $v_2 = (2, 3, -1, 1)$, $v_3 = (1, 1, 2, 4)$

$$a := 1/\sqrt{6} \qquad w_1 := a(1, 2, 0, -1)$$

$$b := 1/\sqrt{246} \qquad w_2 := b(5, 4, -6, 13)$$

$$c := 1/\sqrt{20,295} \qquad w_3 := c(7, 22, 131, 51)$$

Solution:

$$c :=$$

$$1/\sqrt{20,295}$$

$$w_3 := c(7, 22, 131, 51)$$

e.g.: $v_1 = (1, 2, 0, -1)$, $v_2 = (2, 3, -1, 1)$, $v_3 = (1, 1, 2, 4)$

$a := 1/\sqrt{6}$ $\quad$ $w_1 := a(1, 2, 0, -1)$

$b := 1/\sqrt{246}$ $\quad$ $w_2 := b(5, 4, -6, 13)$

$c := 1/\sqrt{20,295}$ $\quad$ $w_3 := c(7, 22, 131, 51)$

Solution:

$$w_1 = (1, 2, 0, -1)/\sqrt{6}$$

$$w_2 = (5, 4, -6, 13)/\sqrt{246}$$

$$w_3 = (7, 22, 131, 51)/\sqrt{20,295} \quad \blacksquare$$

SKILLS:

Find the length of a vector.
Determine if two vectors are orthogonal.
Gram-Schmidt orthonormalization.

Rotations and reflections

Fact: If R is an orthogonal matrix, then either $\det(R) = 1$ or $\det(R) = -1$.

Proof: $[\det(R)]^2 = [\det(R)][\det(R^t)]$
$= \det(RR^t) = \det(I) = 1.$ QED

Definitions: An orthogonal matrix of determinant 1 called a **rotation**.

esp. in three dims sometimes: "special orthogonal"

An orthogonal matrix of determinant -1 called a **reflection**.

esp. in three dims

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal** if $L = L_M$, for some orthogonal $M \in \mathbb{R}^{n\times n}$.

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is a **rotation** if $L = L_M$, for some rotation $M \in \mathbb{R}^{n\times n}$.

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is a **reflection** if $L = L_M$, for some refl. $M \in \mathbb{R}^{n\times n}$.

Fact: Let $n \geq 2$ be an integer. Let $v \in \mathbb{R}^n$.
Assume that $|v| = 1$.
Then $\exists$a rotation $R \in \mathbb{R}^{n \times n}$
s.t. the entries of v are
the entries of the 1st col. of R.

Pf: later . . .

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e.g.: Find a 4×4 rotation whose first column has entries $1/\sqrt{2}$, $1/\sqrt{3}$, 0, $1/\sqrt{6}$.

Either $[v_1 \quad v_2 \quad v_3 \quad v_4]$ or $[v_1 \quad -v_2 \quad v_3 \quad v_4]$ is a rotation matrix.

$[v_1 \quad v_2 \quad v_3 \quad v_4]$ is an orthogonal matrix whose first column "is" v.

Exercise: Apply Gram-Schmidt to v, e_1, e_2, e_3, getting v_1, v_2, v_3, v_4. Note: $v = v_1$

---

v, e_1, e_2, e_3 is a basis of $\mathbb{R}^4$.

WARNING: v, e_1, e_2, e_4 is not a basis of $\mathbb{R}^4$, because e_3 is not in their span.

$$v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{6}} \right)$$

$e_1 = (1, 0, 0, 0)$ $\quad$ $e_3 = (0, 0, 1, 0)$

$e_2 = (0, 1, 0, 0)$ $\quad$ $e_4 = (0, 0, 0, 1)$

---

e.g.: Find a 4×4 rotation whose first column has entries $1/\sqrt{2}$, $1/\sqrt{3}$, 0, $1/\sqrt{6}$.

Fact: Let $n \geq 2$ be an integer. Let $v \in \mathbb{R}^n$.
Assume that $|v| = 1$.
Then $\exists$a rotation $R \in \mathbb{R}^{n \times n}$
s.t. the entries of v are
the entries of the 1st col. of R.

Proof: Let $e_1, \ldots, e_n$ be the std basis of $\mathbb{R}^n$.
Let j be the index of the first
nonzero entry of v.
Then $v, e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$
is a basis of $\mathbb{R}^n$.
Apply Gram-Schmidt,
getting the o.n. basis $v_1, \ldots, v_n$,
with $v = v_1$.
Either $R = [v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_n]$
or $R = [v_1 \quad -v_2 \quad v_3 \quad \cdots \quad v_n]$
works. QED

Fact: Let $n \geq 2$ be an integer. Let $v \in \mathbb{R}^n$.
Assume that $|v| = 1$.
Then $\exists$a rotation $R \in \mathbb{R}^{n \times n}$
s.t. the entries of v are
the entries of the 1st col. of R.

Definition: A **unit vector** is a vector of
length one.

SKILL:

Given a unit vector v in $\mathbb{R}^n$, $n \geq 2$,
make a rotation matrix
whose first column “is” v.

Fact: Let $n \geq 2$ be an integer. Let $v, w \in \mathbb{R}^n$.

Assume that $|v| = 1 = |w|$.

Then $\exists$a rotation $R \in \mathbb{R}^{n \times n}$

s.t. $L_R(v) = w$.

Proof: Let $e_1, \ldots, e_n$ be the std basis of $\mathbb{R}^n$.

Let S be a rotation whose first column "is" v.

Then $L_S(e_1) = v$. Then $L_{S^{-1}}(v) = e_1$.

Let T be a rotation whose first column "is" w.

Then $L_T(e_1) = w$.

Then $L_{TS^{-1}}(v) = (L_T(L_{S^{-1}}(v)) = L_T(e_1) = w$.

Let $R := TS^{-1}$. QED

Fact: Let $n \geq 2$ be an integer. Let $v, w \in \mathbb{R}^n$.

Assume that $|v| = |w|$.

Then $\exists$a rotation $R \in \mathbb{R}^{n \times n}$

s.t. $L_R(v) = w$.

Proof: Let $a := |v| = |w|$.

We may assume that $a \neq 0$.

Let $\quad \check{v} := v/a, \qquad \check{w} := w/a.$

Then $\quad v = a\check{v}, \qquad a\check{w} = w.$

using preceding slide

Choose a rotation $R \in \mathbb{R}^{n \times n}$ s.t. $L_R(\check{v}) = \check{w}$.

Then $L_R(v) = a(L_R(\check{v})) = a\check{w} = w$. QED

SKILL: Given two vectors $v, w \in \mathbb{R}^n$, $n \geq 2$,
of the same length,
make a rotation matrix
that "carries" v to w.

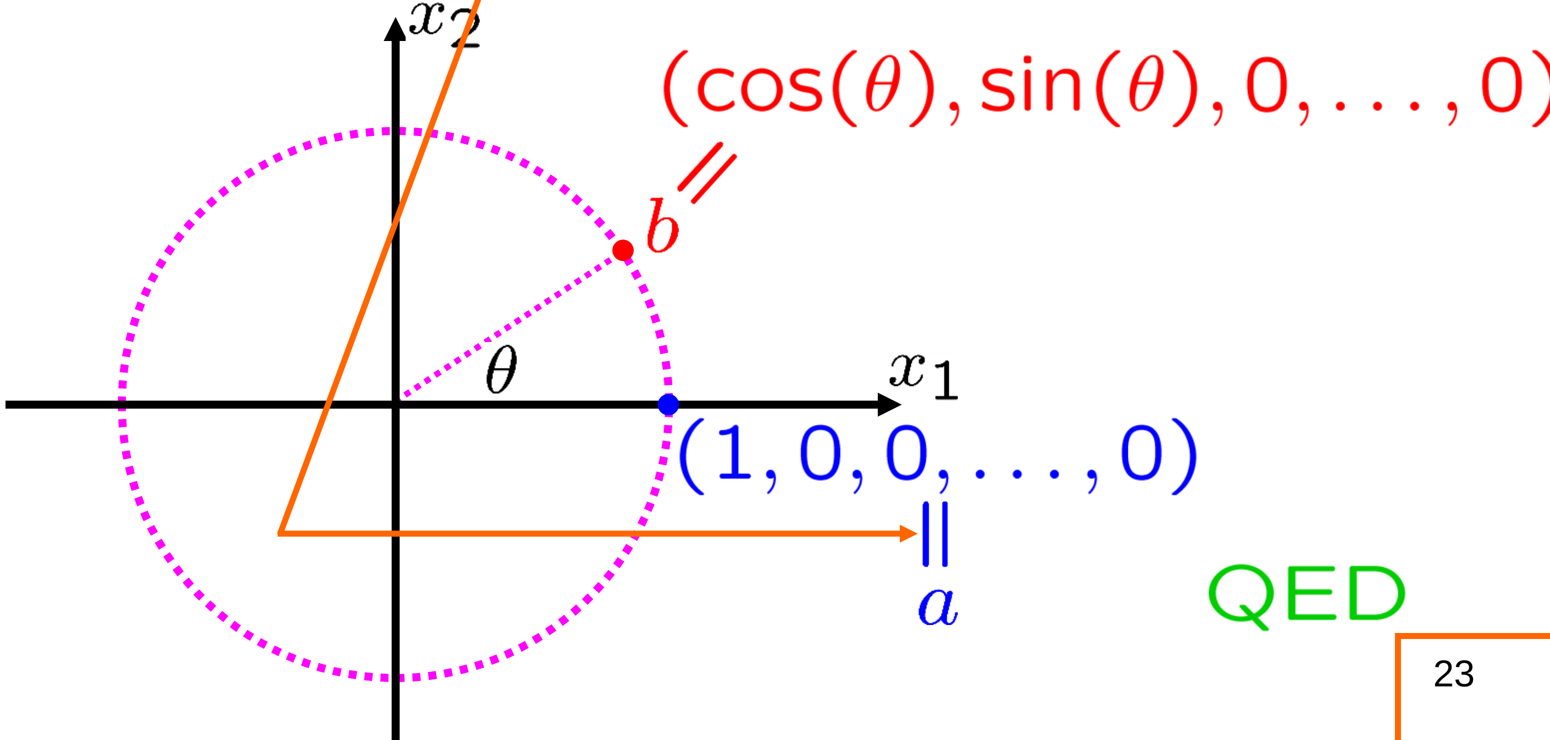
Definition: The **angle** between two vectors $v, w \in \mathbb{R}^n$ is the angle between
the line through 0 and v
and
the line through 0 and w.

SKILL: Given two vectors $v, w \in \mathbb{R}^n$, $n \geq 2$,
of the same length,
make a rotation matrix
that "carries" v to w.

Definition: A **unit vector** is a vector of length one.

Fact: Let a and b be unit vectors.
Assume that $a = (1, 0, 0, \ldots, 0)$
Assume that b is in the x_1, x_2-plane.
Let θ be the angle between a and b.
Then $a \cdot b = \cos(\theta)$.

Proof:



QED

Fact: Let v and w be unit vectors.
Let θ be the angle between v and w.
Then $v \cdot w = \cos(\theta)$.

Here's how to find such an M...

$$Pw^{\text{CV}} = \begin{bmatrix} c \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix}$$ where $u = (u_1, \ldots, u_{n-1})$

$$Qu = \begin{bmatrix} |u| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Mv^{\text{CV}} = ([1] \oplus Q)Pv^{\text{CV}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Mw^{\text{CV}} = ([1] \oplus Q)(Pw^{\text{CV}}) = \begin{bmatrix} c \\ |u| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$M := ([1] \oplus Q)P$$

Pf: Let M be a rotation matrix dot-product-preserving
such that $L_M(v) = (1, 0, 0, \ldots, 0)$
and
such that $L_M(w)$ is in the x_1, x_2-plane.

Let $a := L_M(v)$ and $b := L_M(w)$.
Then $a \cdot b = v \cdot w$. using preceding slide
The angle between a and b is θ.
Then $\cos(\theta) = a \cdot b = v \cdot w$. QED

Fact: Let a and b be vectors.
Let θ be the angle between a and b.
Then $a \cdot b = [|a|][|b|][\cos(\theta)]$.

Proof: If $a = 0$ or $b = 0$, then
$a \cdot b = 0$ and
$[|a|][|b|][\cos(\theta)] = 0$,
so we may assume that $a \neq 0 \neq b$.

Let $v := a/|a|$ and let $w := b/|b|$.
Then $v \cdot w = \cos(\theta)$.

$$a \cdot b = [|a|v] \cdot [|b|w] = [|a|][|b|][v \cdot w]$$
$$= [|a|][|b|] \cos(\theta).$$

QED

Summary:

Defined orthogonal = distance-preserving

same as length-preserving

same as dot-product-preserving

same as (inverse = transpose)

same as orthonormal rows

same as orthonormal columns

Fact: orthogonal = rotation or reflection

Fact: $a \cdot b = (|a|)(|b|)(\cos(\theta))$

Let B be the polarization of a pos. semidef. Q.

$|x| \le y$

iff $-y \le x \le y$

$$|\overbrace{B(v,w)}^{x}| \le \overbrace{\sqrt{Q(v)}\sqrt{Q(w)}}^{y}$$

$$-\sqrt{Q(v)}\sqrt{Q(w)} \le B(v,w) \le \sqrt{Q(v)}\sqrt{Q(w)}$$

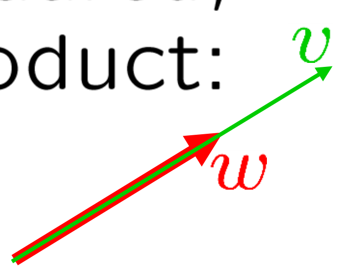
---

v and w are said to be

perfectly correlated (under Q) if

$$B(v,w) = \sqrt{Q(v)}\sqrt{Q(w)}$$

Picture, when Q is length squared,
and B is dot product:



Let B be the polarization of a pos. semidef. Q.

$$|B(v,w)| \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

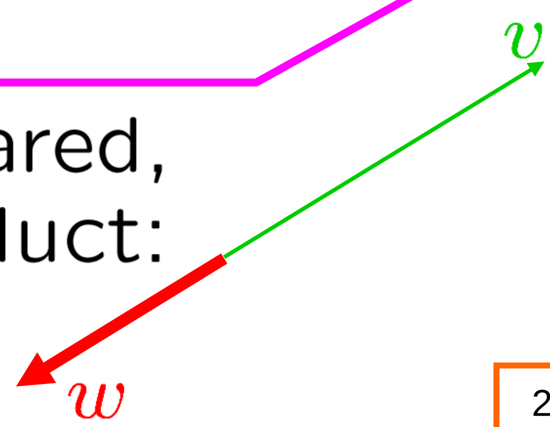
$$-\sqrt{Q(v)}\sqrt{Q(w)} \leq B(v,w) \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

v and w are said to be

perfectly anti-correlated (under Q) if

$$-\sqrt{Q(v)}\sqrt{Q(w)} = B(v,w)$$

Picture, when Q is length squared,
and B is dot product:



Let B be the polarization of a pos. semidef. Q.

$$|B(v,w)| \le \sqrt{Q(v)}\sqrt{Q(w)}$$

$$-\sqrt{Q(v)}\sqrt{Q(w)} \le B(v,w) \le \sqrt{Q(v)}\sqrt{Q(w)}$$

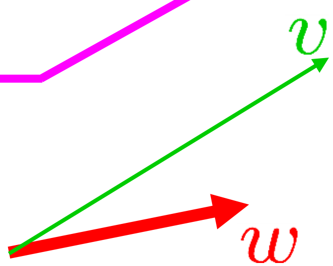
v and w are said to be

positively correlated **(under Q)** if

$$B(v,w) > 0$$

Picture, when Q is length squared,

and B is dot product:



Let B be the polarization of a pos. semidef. Q.

$$|B(v,w)| \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

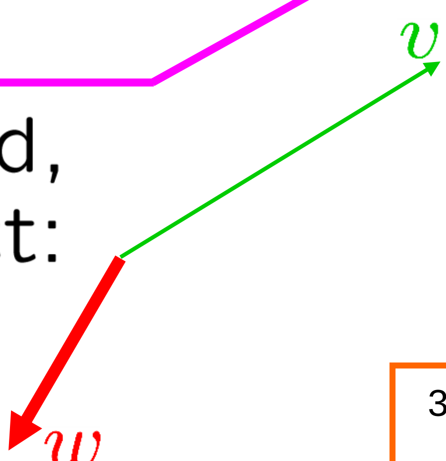
$$-\sqrt{Q(v)}\sqrt{Q(w)} \leq B(v,w) \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

v and w are said to be

negatively correlated (under Q) if

$$B(v,w) < 0$$

Picture, when Q is length squared,
and B is dot product:



Let B be the polarization of a pos. semidef. Q.

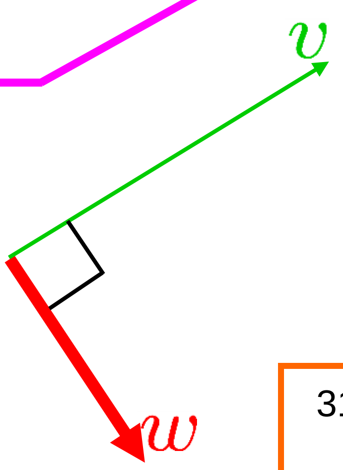
$$|B(v,w)| \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

$$-\sqrt{Q(v)}\sqrt{Q(w)} \leq B(v,w) \leq \sqrt{Q(v)}\sqrt{Q(w)}$$

v and w are said to be **uncorrelated (under Q)** if

$$B(v,w) = 0$$

Picture, when Q is length squared, and B is dot product:



Another BIG IDEA: SPECTRAL THEORY

Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form def'd by

$$Q(x,y) = 3x^2 + \boxed{4xy} + 3y^2$$

Let $D : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form def'd by

$$D(x,y) = 2x^2 + 7y^2$$

$$[Q] = \begin{bmatrix} 3 & \boxed{2} \\ \boxed{2} & 3 \end{bmatrix} \qquad [D] = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

Graphing $\{(x,y) \,|\, D(x,y) = 7.3\}$ is "easy",
because $D(x,y)$ has no "mixed" term.

Def'n: A quadratic form is **diagonal** if its matrix is diagonal.

The SPECTRAL THEOREM: proof later...

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.
Then $\exists$a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$
s.t. $F \circ R : \mathbb{R}^n \to \mathbb{R}$ is diagonal.

Another BIG IDEA: SPECTRAL THEORY

Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form def'd by

$$Q(x,y) = 3x^2 + 4xy + 3y^2$$

---

e.g.: $n = 2$ and $F = Q$

$$R(x,y) = \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right)$$

STOP

$$(Q \circ R)(x,y) = 5x^2 + y^2$$

no mixed "xy" term, *i.e.*, diagonal

Def'n: A quadratic form is **diagonal** if its matrix is diagonal.

The SPECTRAL THEOREM: proof later. . .

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.

Then $\exists$a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$

s.t. $F \circ R : \mathbb{R}^n \to \mathbb{R}$ is diagonal.