Financial Mathematics

Eigenvalues and eigenvectors

The world of Ramagrog

On Ramagrog, there live
Ramatins and Grogali.

Ramatins and Grogali take
one year to mature. No old age death!

Each spring, each Ramatin produces
14 Ramatins and 28 Grogali.
Each fall, each mature Grogalus eats
6 Ramatins and 11 Grogali.

Current population (just before production):
35 Ramatins and 79 Grogali.

Questions:
How many Ramatins and Grogali
one year from now?
and ten years from now?

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How many Ramatins and Grogali
one year from now?

---

35 Ramatins $\longrightarrow \begin{cases} 15 \cdot 35 \text{ Ramatins,} \\ 28 \cdot 35 \text{ Grogali} \end{cases}$

$\underbrace{\text{79 Grogali}}_{\text{mature}}$ $\longrightarrow \begin{cases} -\ 6 \cdot 79 \text{ Ramatins,} \\ -11 \cdot 79 \text{ Grogali} \end{cases}$

$$35\ \text{Ramatins} \longrightarrow \begin{cases} 15 \cdot 35\ \text{Ramatins}, \\ 28 \cdot 35\ \text{Grogali} \end{cases}$$

$$\underbrace{79\ \text{Grogali}}_{\text{mature}} \longrightarrow \begin{cases} -\ 6 \cdot 79\ \text{Ramatins}, \\ -11 \cdot 79\ \text{Grogali} \end{cases}$$

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$$35\ \text{Ramatins} \longrightarrow \begin{cases} 15 \cdot 35\ \text{Ramatins}, \\ 28 \cdot 35\ \text{Grogali} \end{cases}$$

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$$\underbrace{79 \text{ Grogali}}_{\text{mature}} \longrightarrow \begin{cases} -\ 6 \cdot 79 \text{ Ramatins,} \\ -11 \cdot 79 \text{ Grogali} \end{cases}$$

$$\left.\begin{matrix} 35 \text{ Ramatins,} \\ 79 \text{ Grogali} \end{matrix}\right\} \longrightarrow \begin{cases} 15 \cdot 35 - 6 \cdot 79 \text{ Ramatins,} \\ 28 \cdot 35 - 11 \cdot 79 \text{ Grogali} \end{cases}$$

Exercise: Do the arithmetic.

$$\left.\begin{matrix} r_n \text{ Ramatins,} \\ g_n \text{ Grogali} \end{matrix}\right\} \longrightarrow \begin{cases} 15r_n - 6g_n \text{ Ramatins,} \\ 28r_n - 11g_n \text{ Grogali} \end{cases}$$

$$\begin{aligned} r_{n+1} &= 15r_n - 6g_n \\ g_{n+1} &= 28r_n - 11g_n \end{aligned}$$

$$\begin{aligned} r_{n+1} &= 15r_n - 6g_n \\ g_{n+1} &= 28r_n - 11g_n \end{aligned}$$

$$B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$\begin{aligned} r_{n+1} &= 15r_n - 6g_n \\ g_{n+1} &= 28r_n - 11g_n \end{aligned}$$

$$r_{n+1} = 15r_n - 6g_n$$
$$g_{n+1} = 28r_n - 11g_n$$

$$p_{n+1} := \begin{bmatrix} r_{n+1} \\ g_{n+1} \end{bmatrix} \qquad B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$p_n := \begin{bmatrix} r_n \\ g_n \end{bmatrix} \qquad Bp_n = \begin{bmatrix} 15r_n - 6g_n \\ 28r_n - 11g_n \end{bmatrix}$$

$$p_{n+1} = B\,p_n$$

$$p_{n+1} = B\,p_n \qquad B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad p_n = \begin{bmatrix} r_n \\ g_n \end{bmatrix}$$

Current population
(just before production):

35 Ramatins and 79 Grogali.

$$B := \begin{bmatrix} & -6 \\ 28 & -11 \end{bmatrix}$$

$$p_n := \begin{bmatrix} r_n \\ g_n \end{bmatrix}$$

$$p_{n+1} = B\,p_n$$

$p_{n+1} = B\, p_n$	$B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$	$p_n = \begin{bmatrix} r_n \\ g_n \end{bmatrix}$
Current population (just before production): 35 Ramatins and 79 Grogali.		$p_0 = \begin{bmatrix} 35 \\ 79 \end{bmatrix}$

Question: How many Ramatins and Grogali ten years from now?

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = p_{10} = Bp_9 = B^2 p_8 = B^3 p_7 = \cdots$$
$$= B^{10} p_0$$

Exercise: Compute B^2, $(B^2)^2$, $((B^2)^2)^2$.
Compute $B^{10} = [((B^2)^2)^2][B^2]$
Compute $p_{10} = B^{10} p_0$

Cultural note:

Ramatins and Grogali run in herds!

A big herd has 3 Ramatins and 7 Grogali.

A little herd has 1 Ramatin and 2 Grogali.

Current population (just before production):

35 Ramatins and 79 Grogali.

Cultural note:

Current population has

9 big herds and 8 little herds.

$$\begin{bmatrix} 35 \\ 79 \end{bmatrix} = 9 \underbrace{\begin{bmatrix} 3 \\ 7 \end{bmatrix}} + 8 \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\| \qquad \stackrel{..}{\|} \qquad \stackrel{..}{\|}$$

$$p_0 \qquad h^* \qquad h_*$$

$$p_0 = 9 \quad h^* + 8 \quad h_*$$

$$B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad h^* = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad h_* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Cultural note:

A big herd begets a big herd.
A little herd begets three little herds.

Exercise: Do the arithmetic.

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad B\,h^* = h^*$$

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad Bh_* = 3\,h_*$$

Cultural note:

Ramatins and Grogali run in herds!
A big herd has 3 Ramatins and 7 Grogali.
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Cultural note:

Current population has

9 big herds and 8 little herds.

Cultural note:

A big herd begets a big herd.

A little herd begets three little herds.

Population after 10 years has

9 big herds and $8 \cdot 3^{10}$ little herds.

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = 9 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 8 \cdot 3^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = 9 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 8 \cdot 3^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 472,419 \\ 944,847 \end{bmatrix}$$

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = 9 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 8 \cdot 3^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = 9 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 8 \cdot 3^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 472,419 \\ 944,847 \end{bmatrix}$$

$B\,h^* = h^*$	"B fixes h^*."	"h^* is an eigenvector for B with eigenvalue 1."
$Bh_* = 3\,h_*$	"B triples h_*."	"h_* is an eigenvector for B with eigenvalue 3."

$$p_0 = 9 \quad h^* + 8 \quad h_*$$

"p_0 is a linear combination of h^* and h_* with coefficients 9 and 8."

The big idea of eigenvectors and eigenvalues:

If you want to apply a matrix many times to a one-column matrix
(a.k.a. a "column vector"),
the computation becomes much simpler
if you can write the column vector as
a linear combination of eigenvectors.

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e.g.:

Computing $B^{10} p_0$ looks hard,
but becomes much easier if you note that

$$p_0 = 9\, h^* + 8\, h_*$$

$$B\, h^* = h^* \qquad B h_* = 3\, h_*$$

A big herd has 3 Ramatins and 7 Grogali.
A little herd has 1 Ramatin and 2 Grogali.

Transition from big herds/little herds count to Ramatin/Grogali count:

Assume x big herds and y little herds.

Count the number r of Ramatin and the number g of Grogali.

$$r = 3x + y$$
$$g = 7x + 2y$$

$$\begin{bmatrix} r \\ g \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}}_{\substack{\vdots \\ \| \\ C}} \begin{bmatrix} x \\ y \end{bmatrix}$$

C : bh/lh $\to$ R/G

A big herd has 3 Ramatins and 7 Grogali.
A little herd has 1 Ramatin and 2 Grogali.

Transition from Ramatin/Grogali count
to big herds/little herds count:

Assume r Ramatin and g Grogali.

Count the number x of big herds
and the number y of little herds.

$$\begin{bmatrix} r \\ g \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}^{-1}}_{\substack{\| \\ C^{-1}}} \begin{bmatrix} r \\ g \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

C : bh/lh $\rightarrow$ R/G

C^{-1} : R/G $\rightarrow$ bh/lh

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{\displaystyle \overset{\because}{\|} \atop D} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3y \end{bmatrix}$$

D is a diagonal matrix.
D fixes top entry, triples bottom entry.

No such simple description of what

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$ does!

Diagonal matrices are
much easier to understand
and much easier for computations
than non-diagonal matrices.

To apply B to an R/G population count p

change to a bh/lh count (multiply by C^{-1}),

then fix top entry, triple bottom entry (multiply by D),

then change to an R/G count (multiply by C).

$$Bp = CDC^{-1}p \qquad B = CDC^{-1}$$

B is "conjugate" or "similar" to D via (left) C.

B is not diagonal, but is "diagonalizable".

$$C \;:\; \text{bh/lh} \to \text{R/G}$$
$$C^{-1} :\; \text{R/G} \to \text{bh/lh}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad B^{10} = \ ?????$$

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \overbrace{\begin{bmatrix} 3 \\ 7 \end{bmatrix}}^{h^*} = 1 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

eigenvalue

eigenvector

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{h_*} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eigenvalue

eigenvector

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad B^{10} = ?????$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

eigenvector eigenvector

eigenvalue eigenvalue

eigenvalue eigenvector

$$1 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eigenvalue eigenvector

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad B^{10} = ?????$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = CDC^{-1} \quad ????? \qquad C^{-1} = ?????$$

$$\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2 \times \text{R1} \to \text{R2}} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{R1} \leftrightarrow \text{R2}} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{-3 \times \text{R1} \to \text{R2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$B^{10} = \text{?????}$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = CDC^{-1} \quad \text{?????}$$

$$C^{-1} =$$

$$CD = \begin{bmatrix} 3 & 3 \\ 7 & 6 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$B^{10} = \text{?????}$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = CDC^{-1} \quad \text{?????}$$

$$CD = \begin{bmatrix} 3 & 3 \\ 7 & 6 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$$

$$CDC^{-1} = \begin{bmatrix} 3 & 3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} = B$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad B^{10} = ?????$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = CDC^{-1}$$

B is "conjugate" or "similar" to D via (left) C.

B is not diagonal, but is "diagonalizable".

$$B^2 = (CD\cancel{C^{-1}})(\cancel{C}DC^{-1}) = CD^2C^{-1}$$

$$B^{10} = (CDC^{-1}) \cdots (CDC^{-1}) = CD^{10}C^{-1}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad B^{10} = ?????$$

$$C = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$D^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{10} \end{bmatrix}$$

$$B^{10} = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix} = \text{Exercise}$$

$$B^2 = (CD\cancel{C^{-1}})(\cancel{C}DC^{-1}) = CD^2C^{-1}$$

$$B^{10} = (CDC^{-1}) \cdots (CDC^{-1}) = CD^{10}C^{-1}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Questions:

How to find C, D so that $B = CDC^{-1}$, with D diagonal?

I.e.: How to diagonalize B?

I.e.: How to find eigvects & eigvals of B?

Are all square matrices diagonalizable?

How to identify and diagonalize the matrices that are diagonalizable?

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: tenth power

Question:
Are all square matrices diagonalizable?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Say PMP^{-1} is diagonal.

WGot: Contradiction.

$$PMP^{-1} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$PMP^{-1} = 0$$
$$M = P^{-1} 0 P = 0$$

Contradiction.

$$(PMP^{-1})^2 = PM^2P^{-1} = P0P^{-1} = 0$$
$$\|$$
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

$$a^2 = 0 = b^2$$
$$a = 0 = b$$

Question:

Are all square matrices diagonalizable?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Say PMP^{-1} is diagonal.

Got: Contradiction.

Definition:

A (square) matrix is **nilpotent** if some power of it is zero.

e.g.: M, any strictly upper triangular, any conjugate of a str. upper triangular

Fact: Any nilpotent diagonalizable matrix is 0.

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e.g.: M, any strictly upper triangular,
any conjugate of a str. upper triangular

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

λ is an eigenvalue of B

iff $\exists$a vector $v \neq 0$ s.t. $Bv = \lambda v$

iff $\exists$a vector $v \neq 0$ s.t. $(B - \lambda I)v = 0$

iff $\det(B - \lambda I) = 0$

iff $\det \begin{bmatrix} 15 - \lambda & -6 \\ 28 & -11 - \lambda \end{bmatrix} = 0$

iff $(15 - \lambda)(-11 - \lambda) - (-6)(28) = 0$

iff $\lambda^2 - 4\lambda + 3 = 0$

iff $\lambda = 1$ or $\lambda = 3$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

λ is an eigenvalue of B

iff $\lambda = 1$ or $\lambda = 3$

The eigenvalues of B are 1 and 3.

iff $\lambda = 1$ or $\lambda = 3$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

λ is an eigenvalue of B
iff $\lambda = 1$ or $\lambda = 3$

The eigenvalues of B are 1 and 3.

Subquestion: How to find the eigenvectors of B with eigenvalue 1?

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

v is a 3-eigenvector of B

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

eigenvector with eigenvalue 3

v is a 3-eigenvector of B

iff $Bv = 3v$ and $v \neq 0$

iff $(B - 3I)v = 0$ and $v \neq 0$

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

v is a 3-eigenvector of B

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

v is a 3-eigenvector of B

$$B - 3I = \begin{bmatrix} 12 & -6 \\ 28 & -14 \end{bmatrix}$$

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

v is a 3-eigenvector of B

iff $v \in [\ker(B - 3I)] \setminus \{0\}$

$$B - 3I = \begin{bmatrix} 12 & -6 \\ 28 & -14 \end{bmatrix} \xrightarrow[(1/14) \times R2]{(1/6) \times R1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{iff} \quad \begin{bmatrix} 2x - y \\ 2x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

iff $y = 2x$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

v is a 3-eigenvector of B

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

$$B - 3I = \begin{bmatrix} 12 & -6 \\ 28 & -14 \end{bmatrix} \xrightarrow[(1/14)\times \text{R2}]{(1/6)\times \text{R1}} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ iff $y = 2x$

$$\ker(B - 3I) = \ker \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

iff $y = 2x$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3?

v is a 3-eigenvector of B

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

$$B - 3I = \begin{bmatrix} 12 & -6 \\ 28 & -14 \end{bmatrix} \xrightarrow[(1/14) \times R2]{(1/6) \times R1} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{iff} \quad y = 2x$$

$$\ker(B - 3I) = \ker \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \middle| x \in \mathbb{R} \right\}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 3? ☺

v is a 3-eigenvector of B

iff $v \in [\ker(B - 3I)] \backslash \{0\}$

iff v is a nonzero multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\ker(B - 3I) = \ker \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Subquestion: How to find the eigenvectors of B with eigenvalue 1? ☺

$$B - I = \begin{bmatrix} 14 & -6 \\ 28 & -12 \end{bmatrix} \xrightarrow[(1/4)\times \text{R2}]{(1/2)\times \text{R1}} \begin{bmatrix} 7 & -3 \\ 7 & -3 \end{bmatrix}$$

$$\ker(B - I) = \left\{ \begin{bmatrix} 3x \\ 7x \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$$

v is a 1-eigenvector of B

iff v is a nonzero multiple of $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

The eigenvalues of B are 1 and 3.

v is a 3-eigenvector of B

iff v is a nonzero multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

v is a 1-eigenvector of B

iff v is a nonzero multiple of $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

The eigenvalues of B are 1 and 3.

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Question: How to diagonalize B?

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

Question: How to find the eigenvectors & eigenvalues of B?

The eigenvalues of B are 1 and 3.

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Question: How to diagonalize B?

$$\begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longleftarrow 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longleftarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longleftarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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square matrix

Definition:

The **characteristic polynomial of** M

is $\chi_M(\lambda) := \det[M - \lambda I]$.

Note: It is an expression of λ; in fact,
it is a *polynomial* in λ.
The underlying function χ_M is
a polynomial function,
sometimes also called the
characteristic polynomial of M.

square matrix

Definition:

The **characteristic polynomial of** M is $\chi_M(\lambda) := \det[M - \lambda I]$.

Definition:

r is an **eigenvalue of** M means that $\chi_M(r) = 0$.

scalar

I.e.: $\det(M - rI) = 0$.

I.e.: $M - rI$ is *not* invertible.

I.e.: $\ker(M - rI) \neq \{0\}$.

I.e.: $\exists$column vector $v \neq 0$ s.t. $(M - rI)v = 0$.

I.e.: $\exists$column vector $v \neq 0$ s.t. $Mv = rv$.

Definition:

r is an **eigenvalue of** M
means that $\chi_M(r) = 0$.

I.e.: $\exists$column vector $v \neq 0$ s.t.

M simply multiplies v by r → $Mv = rv$.

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M simply multiplies v by r → $Mv = rv$.

Definition:

The r-**eigenspace** **of** M is $\ker(M - rI)$.

Definition:

v is an r-**eigenvector** **of** M
means that $v \in [\ker(M - rI)] \backslash \{0\}$.

I.e.: $(M - rI)v = 0$ and $v \neq 0$.
I.e.: $Mv = rv$ and $v \neq 0$.
I.e.: v is a nonzero element of the r-eigenspace of M.

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Note: 0 is *never* an eigenvector, but is in *every* eigenspace.

SKILL:

Given a square matrix,

find its eigenvalues and

for each eigenvalue r,

find a basis of the r-eigenspace.

STOP