Financial Mathematics

Diagonalization of matrices

Question: How to identify and diagonalize the matrices that are diagonalizable?

Note: Real diagonalizable and complex diagonalizable are different.

e.g.: $M := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\qquad M^2 = -I$

Suppose $PMP^{-1} := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = (PMP^{-1})^2 = PM^2P^{-1}$$

$$= P(-I)P^{-1} = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$a^2 = -1 = b^2$, so no *real* solution.

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$i = \sqrt{-1}$

$C := \begin{bmatrix} \boxed{\begin{matrix} 1-i \\ 1+i \end{matrix}} & \begin{matrix} 1+i \\ 1-i \end{matrix} \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+i \end{bmatrix} = \boxed{i\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \begin{bmatrix} 1-i \\ -1-i \end{bmatrix} = -i\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

$$C^{-1}M\boxed{C\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = C^{-1}M\boxed{\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}} = C^{-1}\boxed{i\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}}$$

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$$C^{-1}MC\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C^{-1}M\begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = \overset{\longleftrightarrow}{i\,C^{-1}}\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}$$

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$$C^{-1}MC \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \; C^{-1} \boxed{\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}} = i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$C^{-1}MC\begin{bmatrix} 1 \\ 0 \end{bmatrix} = i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C^{-1}MC = \left[i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right.$$

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$$C^{-1}MC = \begin{bmatrix} \boxed{\begin{matrix} i \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -i \end{matrix}} \end{bmatrix}$$

$$C^{-1}M^{10}C = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{10} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$C^{-1}MC\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{i\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \qquad C^{-1}MC\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{-i\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

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$$C^{-1}MC = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$C^{-1}M^{10}C = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{10} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M^{10} = C\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}C^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The diagonalization algorithm

Say $\chi_M(\lambda) = (r_1 - \lambda)^{e_1} \cdots (r_k - \lambda)^{e_k}$,
with $r_1, \ldots, r_k$ distinct complex numbers.

The char. poly. of M has roots $r_1, \ldots, r_k$,
with multiplicities $e_1, \ldots, e_k$.

Definition:
If r is a root of a polynomial in λ,
then the multiplicity of r is the largest e
such that $(r - \lambda)^e$ divides the polynomial.

e.g.: The roots of $(1 - \lambda)^4 (3 - \lambda)^7 (10 + \lambda)$

are:	1,	3,	−10.
Multiplicities are:	4,	7,	1.

The diagonalization algorithm

Say $\chi_M(\lambda) = (r_1 - \lambda)^{e_1} \cdots (r_k - \lambda)^{e_k}$,
with $r_1, \ldots, r_k$ distinct complex numbers.

Suppose, for each integer $j \in [1, k]$,
that the eigenspace $E_j := \ker(M - r_j I)$
satisfies $e_j = \dim E_j$.

Write out a basis for E_1, then one for E_2,
etc., until you list a basis for E_k.

The full list has $e_1 + \cdots + e_k$ column vectors.

Make these into a matrix C
with $e_1 + \cdots + e_k$ columns.

Then $C^{-1}MC$ will be diagonal with entries

$$\underbrace{r_1, \ldots, r_1,}_{e_1 \text{ times}} \quad , \quad \underbrace{r_2, \ldots, r_2}_{e_2 \text{ times}} \quad , \ldots , \quad \underbrace{r_k, \ldots, r_k}_{e_k \text{ times}}.$$

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

Subquestion: Why doesn't the algorithm from the last slide work on the following matrix?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\chi_M(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

$$= (-\lambda)^2$$

$$= \lambda^2$$

$$= (0 - \lambda)^2$$

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

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$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\chi_M(\lambda) = (0 - \lambda)^2$$

$$\chi_M(\lambda)$$

$$\chi_M(\lambda) = (r_1 - \lambda)^{e_1} \cdots (r_k - \lambda)^{e_k}$$

$$= (0 - \lambda)^2$$

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

Subquestion: Why doesn't the algorithm from the last slide work on the following matrix?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\chi_M(\lambda) = (0 - \lambda)^2$$

$$\boxed{k = 1 \quad r_1 = 0 \quad e_1 = 2}$$

Suppose $\boxed{r_1 = 0}$

that the eigenspace $E_1 := \ker(M - r_1 I)$ satisfies $e_1 = \dim E_1$. $\boxed{e_1 = 2}$

$$\chi_M(\lambda) = (r_1 - \lambda)^{e_1} \cdots (r_k - \lambda)^{e_k}$$

Suppose, for each integer $j \in [1, k]$, $\boxed{k = 1}$

that the eigenspace $E_j := \ker(M - r_j I)$ satisfies $e_j = \dim E_j$.

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

Subquestion: Why doesn't the algorithm from the last slide work on the following matrix?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\chi_M(\lambda) = (0 - \lambda)^2$$

$$\boxed{k = 1 \quad r_1 = 0 \quad e_1 = 2}$$

Suppose $\boxed{r_1 = 0}$

that the eigenspace $E_1 := \ker(M)$ satisfies $2 = \dim E_1$. $\boxed{e_1 = 2}$

$$\chi_M(\lambda) = (r_1 - \lambda)^{e_1} \cdots (r_k - \lambda)^{e_k}$$

Suppose, for each integer $j \in [1, k]$, $\boxed{k = 1}$

that the eigenspace $E_j := \ker(M - r_j I)$ satisfies $e_j = \dim E_j$.

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

Subquestion: Why doesn't the algorithm from the last slide work on the following matrix?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \chi_M(\lambda) = (0 - \lambda)^2$$

$$\boxed{k = 1 \quad r_1 = 0 \quad e_1 = 2}$$

Suppose $\boxed{r_1 = 0}$

that the eigenspace $E_1 := \ker(M)$

satisfies ~~$2 = \dim E_1$~~. $\boxed{e_1 = 2}$ $\boxed{\dim E_1 = 1}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{iff} \quad \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{iff} \quad y = 0$$

$\ker(M) = \begin{bmatrix} * \\ 0 \end{bmatrix}$ | Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ | Dimension: 1

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

Subquestion: Why doesn't the algorithm from the last slide work on the following matrix?

$$M := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Answer to subquestion:

Because the dimension of the 0-eigenspace is not equal to the multiplicity of the root 0 in the characteristic polynomial.

Fact: A matrix is diagonalizable iff for any root of the characteristic polynomial, its multiplicity is equal to the dimension of its eigenspace.

Question: How to identify and diagonalize the matrices that are complex diagonalizable?

SKILL:

Given a matrix, use the fact below to determine if it is diagonalizable. If it is, use the diagonalization algorithm to diagonalize it.

Fact: A matrix is diagonalizable iff for any root of the characteristic polynomial, its multiplicity is equal to the dimension of its eigenspace.

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: tenth power

SEVERAL NON-DIAGONALIZABLE MATRICES

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Exercise: Show that, in char poly, the root 2 has multiplicty 3 but the 2-eigenspace has dimension 1.

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$$\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}}_{9I} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: A matrix is scalar if it is a scalar multiple of the identity.

e.g.: $\underset{\text{scalar}}{9}\underset{\text{identity}}{I}$

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$$\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\substack{\| \\ \ddot{N}}}$$

Definition: A matrix is standard nilpotent if it's square and has 1s just above the diagonal and 0s everywhere else.

e.g.: N

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e.g.: tenth power

$$\underbrace{\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}}_{\substack{\| \\ \cdot\cdot \\ B}} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: A matrix is a **Jordan block** if it's the sum of a scalar matrix and a standard nilpotent matrix.

e.g.: B, $[4]$

Note: A J. block is diagonalizable iff it's 1×1.

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: tenth power

Def'n: Let A be $m \times m'$ and let B be $n \times n'$. Then $\boxed{A \oplus B}$ is the $(m+n) \times (m'+n')$ matrix with A in the upper left, with B in the lower right and with 0s elsewhere.

the **direct sum of A and B**

e.g.:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \oplus \begin{bmatrix} g & h \end{bmatrix}$$

2×3

1×2

$$\|$$

$$\begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ 0 & 0 & 0 & g & h \end{bmatrix}$$

$(2+1) \times (3+2)$
or
3×5

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the **direct sum** **of** A **and** B

$$A \oplus B$$

$m \times m'$

$n \times n'$

$$\|$$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$(m+n) \times (m'+n')$

Last question

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the **direct sum** **of A and B**

Fact: Let $A_1, \ldots, A_l$ be square matrices.
Then $A_1 \oplus \cdots \oplus A_l$ is diagonalizable iff $\forall$integers $j \in [1, l]$, A_j is diagonalizable.

Jordan canonical form

Every (complex) matrix is (complex) conjugate to a direct sum of (one or more) Jordan blocks.

A direct sum of J. blocks is diagonalizable iff all blocks 1×1 iff diagonal

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e.g.: tenth power

Def'n: Let A be $m \times m'$ and let B be $n \times n'$.
Then $\boxed{A \oplus B}$ is the $(m+n) \times (m'+n')$ matrix
with A in the upper left,
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and with 0s elsewhere.

the **direct sum of** A **and** B

Fact: Let $A_1, \ldots, A_l$ be square matrices.
Then $A_1 \oplus \cdots \oplus A_l$ is diagonalizable
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subQ: Is J. canon. form "computationally good"?

e.g.: tenth power

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: tenth power

$$\underbrace{\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}}_{8I+N} = 8\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

$$(8I+N)^{10} =$$

Note: Scalar matrices commute with all matrices.

$$8^{10}I + 10(8^9N) + (10 \cdot 9/2)(8^8N^2) + \cdots$$

WARNING: Binomial expansion of $(A+B)^n$ is usually wrong *unless* A and B commute.

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$$(8I+N)^{10} =$$

$$8^{10}I + 10(8^9N) + (10\cdot 9/2)(8^8\underset{\substack{\| \\ ?????}}{\boxed{N^2}}) + \cdots$$

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$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = 0$$

$$N^4 = 0$$

etc.

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e.g.: tenth power

$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}^{10} =$$

$$8^{10}I + 10(8^9N) + \overbrace{(10 \cdot 9/2)}^{45}(8^8N^2) \overbrace{+ \cdots}^{0}$$

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}^{10} =$$

$$8^{10}I + 10(8^9N) + (45)(8^8N^2)$$

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$N^3 = 0$

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$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}^{10} = 8^{10} I + 10(8^9 N) + (45)(8^8 N^2)$$

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}^{10} = 8^{10}I + 10(8^9 N) + (45)(8^8 N^2)$$

$$= \begin{bmatrix} 8^{10} & 0 & 0 \\ 0 & 8^{10} & 0 \\ 0 & 0 & 8^{10} \end{bmatrix} + \begin{bmatrix} 0 & 10 \cdot 8^9 & 0 \\ 0 & 0 & 10 \cdot 8^9 \\ 0 & 0 & 0 \end{bmatrix} + \cdots$$

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$$= \begin{bmatrix} 8^{10} & 10 \cdot 8^9 & 0 \\ 0 & 8^{10} & 10 \cdot 8^9 \\ 0 & 0 & 8^{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 45 \cdot 8^8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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e.g.: tenth power

$$\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}^{10} = 8^{10} I + 10(8^9 N) + (45)(8^8 N^2)$$

$$= \begin{bmatrix} 8^{10} & 10 \cdot 8^9 & 45 \cdot 8^8 \\ 0 & 8^{10} & 10 \cdot 8^9 \\ 0 & 0 & 8^{10} \end{bmatrix} = \text{Exercise}$$

Jordan blocks are "computationally good"!!

subQ: Is J. canon. form "computationally good"?

e.g.: tenth power

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

Definition: $\forall$square matrices M,

$$\boxed{e^M} := \boxed{\exp(M)} := \lim_{n\to\infty}[I + (M/n)]^n$$

$$= I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$$

$$e^x := \exp(x) := \lim_{n\to\infty}[1 + (x/n)]^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

subQ: Is J. canon. form "computationally good"?

e.g.:*e.g.*: expower

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

Definition: $\forall$square matrices M,

$$\boxed{e^M} := \boxed{\text{exp}(M)} := \lim_{n\to\infty} [I + (M/n)]^n$$

$$= I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$$

$$\text{exp}\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = ?????$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad N := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} = ?????$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

Fact: If $AB = BA$,

then $e^{A+B} = e^{A}e^{B}$

$$\begin{tikzcd} & e^{9I}e^{N} \\ \exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} \arrow[ur, equal] \arrow[r, equal] & ????? \end{tikzcd}$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

$$e^{9I} = I + 9I + \frac{9^2 I}{2!} + \frac{9^3 I}{3!} + \cdots$$

$$= \left(1 + 9 + \frac{9^2}{2!} + \frac{9^3}{3!} + \cdots\right) I = e^9 I$$

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} \begin{array}{l} = e^{9I} e^{N} \\ = ????? \end{array}$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

$$e^{9I} = \exp\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} e^9 & 0 & 0 & 0 \\ 0 & e^9 & 0 & 0 \\ 0 & 0 & e^9 & 0 \\ 0 & 0 & 0 & e^9 \end{bmatrix}$$

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} \begin{array}{l} = e^{9I}e^{N} \\ = ????? \end{array}$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a “computationally good form”, even for those matrices that are not diagonalizable?

e.g.: exp

$$e^{N} = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \frac{N^4}{4!} + \cdots$$

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} \begin{array}{l} = e^{9I}e^{N} \\ = ????? \end{array}$$

subQ: Is J. canon. form “computationally good”?

e.g.: exp

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \parallel \\ N^5 \\ \quad \rotatebox{-45}{$=$} \\ \qquad N^6 = \cdots \end{array}$$

$$e^N = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \frac{N^4}{4!} + \cdots$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \parallel \\ N^5 \\ \quad \parallel \\ \qquad N^6 = \cdots \end{array}$$

$$e^N = I + N + \frac{N^2}{2!} + \frac{N^3}{3!}$$

$$2! = 2 \cdot 1 = 2 \qquad 3! = 3 \cdot 2 \cdot 1 = 6$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e^N = I + N + \frac{N^2}{2} + \frac{N^3}{6} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} = ?????$$

$$e^{9I+N} = e^{9I}e^{N}$$

$$e^{9I}e^{N} = \begin{bmatrix} e^9 & 0 & 0 & 0 \\ 0 & e^9 & 0 & 0 \\ 0 & 0 & e^9 & 0 \\ 0 & 0 & 0 & e^9 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

e.g.: exp

$$\begin{bmatrix} e^9 & e^9 & e^9/2 & e^9/6 \\ 0 & e^9 & e^9 & e^9/2 \\ 0 & 0 & e^9 & e^9 \\ 0 & 0 & 0 & e^9 \end{bmatrix}$$

$$\|$$

$$e^{9I}e^{N}$$

$$//$$

$$\exp\begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} = e^{9I+N} = ?????$$

Jordan blocks are "computationally good"!!

subQ: Is J. canon. form "computationally good"?

e.g.: exp

Last question

Can we find a "computationally good form", even for those matrices that are not diagonalizable?

Jordan blocks are computationally good!!

$$(A \oplus B)^{10} = (A^{10}) \oplus (B^{10})$$
$$(A \oplus B \oplus C)^{10} = (A^{10}) \oplus (B^{10}) \oplus (C^{10})$$

etc.

(Assuming A, B, C, *etc.* are square matrices.)

$$\exp(A \oplus B) = (\exp(A)) \oplus (\exp(B))$$
$$\exp(A \oplus B \oplus C) = (\exp(A)) \oplus (\exp(B)) \oplus (\exp(C))$$

etc.

(Assuming A, B, C, *etc.* are square matrices.)

Jordan canonical form is computationally good!!

subQ: Is J. canon. form "computationally good"?

e.g.: exp

YES!!

Problem: $\exp \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$

Problem: $\exp \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1} \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix}$$

Problem: $\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1} \left(\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}\right) \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \exp\begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix} = \begin{bmatrix} e^{6i} & 0 \\ 0 & e^{-6i} \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1} \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix}$$

Problem: $\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1} \left(\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} \right) \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \exp\begin{bmatrix} 6i & 0 \\ 0 & -6i \end{bmatrix} = \begin{bmatrix} e^{6i} & 0 \\ 0 & e^{-6i} \end{bmatrix}$$

$$\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} e^{6i} & 0 \\ 0 & e^{-6i} \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1}$$

Problem: $\exp\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}$

$$=\underbrace{\begin{bmatrix}1-i & 1+i\\ 1+i & 1-i\end{bmatrix}\begin{bmatrix}e^{6i} & 0\\ 0 & e^{-6i}\end{bmatrix}}_{/\!/}\underbrace{\begin{bmatrix}1-i & 1+i\\ 1+i & 1-i\end{bmatrix}^{-1}}_{/\!/}$$

$$=\overbrace{\begin{bmatrix}(1-i)e^{6i} & (1+i)e^{-6i}\\ (1+i)e^{6i} & (1-i)e^{-6i}\end{bmatrix}}\overbrace{\begin{bmatrix}1-i & -1-i\\ -1-i & 1-i\end{bmatrix}\frac{-1}{4i}}$$

$$=\begin{bmatrix}-4i\cos 6 & -4i\sin 6\\ 4i\sin 6 & -4i\cos 6\end{bmatrix}\frac{-1}{4i}$$

$e^{6i}=\cos 6+i\sin 6$ $e^{-6i}=\cos 6-i\sin 6$

Problem: $\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} e^{6i} & 0 \\ 0 & e^{-6i} \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (1-i)e^{6i} & (1+i)e^{-6i} \\ (1+i)e^{6i} & (1-i)e^{-6i} \end{bmatrix} \begin{bmatrix} 1-i & -1-i \\ -1-i & 1-i \end{bmatrix} \frac{-1}{4i}$$

$$= \begin{bmatrix} -4i\cos 6 & -4i\sin 6 \\ 4i\sin 6 & -4i\cos 6 \end{bmatrix} \frac{-1}{4i}$$

$$\exp\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{bmatrix}$$

Problem: $\exp \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$

Sol'n: $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

Problem: $\exp \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$

Sol'n: $\begin{bmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{bmatrix}$

Exercise: Compute the first five terms of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}^2 + \frac{1}{3!}\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}^3 + \cdots$$

Add up their $(1,1)$ entries and verify that the result is $1+0-(t^2/(2!))+0+(t^4/(4!))$, which is the start of the power series of $\cos t$.

$$\exp \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{bmatrix}$$

STOP