Financial Mathematics

The Spectral Theorem

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Rotations and reflections

Definition: An **orthogonal matrix** is a matrix $R \in \mathbb{R}^{n \times n}$ s.t. $RR^t = R^t R = I$.

real square

I.e.: R is distance-preserving

I.e.: R is dot-product-preserving

Rotations and reflections

Fact: If R is an orthogonal matrix, then either $\det(R) = 1$ or $\det(R) = -1$.

Proof: $[\det(R)]^2 = [\det(R)][\det(R^t)]$
$= \det(RR^t) = \det(I) = 1$. QED

Definitions: An orthogonal matrix of determinant 1 called a **rotation**.

An orthogonal matrix of determinant -1 called a **reflection**.

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal** if $L = L_M$ for some orthogonal $M \in \mathbb{R}^{n\times n}$.

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is a **rotation** if $L = L_M$ for some rotation $M \in \mathbb{R}^{n\times n}$.

A lin. transf. $L : \mathbb{R}^n \to \mathbb{R}^n$ is a **reflection** if $L = L_M$ for some reflection $M \in \mathbb{R}^{n\times n}$.

Change of variables for quadratic forms

Definition:

For any symmetric matrix $S \in \mathbb{R}^n$,
the quadratic form $\boxed{Q_S} : \mathbb{R}^n \to \mathbb{R}$ is def'd by

$$Q_S(v) = (L_S(v)) \cdot v.$$

Polarization: $\forall$quadratic forms $Q : \mathbb{R}^n \to \mathbb{R}$,
$\exists$unique symmetric $S \in \mathbb{R}^{n \times n}$ s.t. $Q = Q_S$.

Def'n: Quadratic forms $Q, Q' : \mathbb{R}^n \to \mathbb{R}$ are **equivalent** if $\exists$invertible linear $L : \mathbb{R}^n \to \mathbb{R}^n$
s.t. $Q' = Q \circ L$.

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Fact: Let $X, S \in \mathbb{R}^{n \times n}$.
Assume S is symmetric.
Then $Q_S \circ L_X = Q_{X^t S X}$.

motivation,
then pf

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The idea: Quadratic forms are equivalent
iff their matrices are "t-equivalent".

Definition: $S, S' \in \mathbb{R}^{n \times n}$ are t-equivalent
if $\exists$invertible $X \in \mathbb{R}^{n \times n}$ s.t. $S' = X^t S X$.

"S and S' are t-equivalent via X (on the right)"

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the quadratic form $\boxed{Q_S} : \mathbb{R}^n \to \mathbb{R}$ is def'd by

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Fact: Let $X, S \in \mathbb{R}^{n\times n}$.
Assume S is symmetric.
Then $Q_S \circ L_X = Q_{X^tSX}$.

replace S by X^tSX

replace v by $L_X(v)$

motivation, then pf

Proof: $(Q_S \circ L_X)(v) = Q_S(L_X(v))$

$$= [L_S(L_X(v))] \cdot [L_X(v)]$$

$$= [L_{X^t}(L_S(L_X(v))] \cdot v$$

$$= [L_{X^tSX}(v)] \cdot v$$

$$\stackrel{?}{=} Q_{X^tSX}(v)$$

QED

Change of variables for quadratic forms

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Fact: Let $X, S \in \mathbb{R}^{n \times n}$.

Assume S is symmetric.

Then $Q_S \circ L_X = Q_{X^t S X}$.

SKILL: Given S and X,

produce a matrix N such that

$$Q_S \circ L_X = Q_N.$$

WARNING:

$N = X^t S X$, not $X S X^t$.

Diagonal quadratic forms

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the quadratic form $\boxed{Q_S} : \mathbb{R}^n \to \mathbb{R}$ is def'd by

$$Q_S(v) = (L_S(v)) \cdot v.$$

Polarization: $\forall$quadratic forms $Q : \mathbb{R}^n \to \mathbb{R}$,
$\exists$unique symmetric $S \in \mathbb{R}^{n \times n}$ s.t. $Q = Q_S$.

Definition: A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$,
is **diagonal** if $\exists$a diagonal matrix $D \in \mathbb{R}^{n \times n}$
(easily studied) s.t. $Q = Q_D$.

$$D = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \begin{array}{ccc} x & y & z \end{array} & \\ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array} \quad \Rightarrow$$

No cross-terms!

$$Q_D(x, y, z) = ax^2 + by^2 + cz^2$$

Diagonal quadratic forms

e.g.: $Q(w,x,y,z) = 5w^2 - 2x^2 + 4y^2 + \pi z^2$

$$Q(x,y) = 4x^2 + 2y^2$$

non-e.g.: $Q(x,y) = 4x^2 + 2y^2 - xy$

$$Q(w,x,y,z) = 5w^2 - 2x^2 + 4y^2 + \pi wz$$

matrix has $\pi/2$ in the $(1,3)$ and $(3,1)$ entries

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matrix has $\pi/2$ in the $(1,3)$ and $(3,1)$ entries

SKILL:
Recognize whether a given quadratic form is diagonal.

Definition:
A matrix M is **rotationally diagonalizable**
if $\exists$rotation C s.t. $C^t MC$ is diagonal and real.

I.e.: M is t-equivalent to a diagonal matrix
via a rotation matrix (on the right).

I.e.: $\exists$rotation C s.t. $C^{-1}MC$ is diagonal.

I.e.: M is conjugate to a diagonal matrix
via a rotation matrix (on the right).

Goal:
Rotationally diagonalizable = symmetric

(To prove this, we'll need some prerequisites.)

Fact: The eigenvalues of a symmetric (real) matrix are always real.

Proof: Let P be a symmetric (real) matrix,
let $z \in \mathbb{C}$, let $v \in \mathbb{C}^{n\times 1}\backslash\{0\}$
and assume that $Pv = zv$.

Want: $z \in \mathbb{R}$ Want: $z = \bar{z}$

$v \neq 0$, so $v \cdot \overline{v} \neq 0$. dot product in $\mathbb{C}^{n\times 1}$

$$P\overline{v} = \overline{P}\,\overline{v} = \overline{Pv} = \overline{zv} = \bar{z}\overline{v}$$

$$z(v \cdot \overline{v}) = (zv) \cdot \overline{v} = (Pv) \cdot \overline{v}$$
$$= v \cdot (P\overline{v}) = v \cdot (\bar{z}\overline{v}) = \bar{z}(v \cdot \overline{v})$$

$$z\cancel{(v \cdot \overline{v})} = \bar{z}\cancel{(v \cdot \overline{v})}$$

$$z = \bar{z}$$ QED

Fact: If $A, B \in \mathbb{R}^{n\times n}$ are conjugate,
then A and B have
the same characteristic polynomial,
and the same eigenvalues.

Proof: Choose an invertible $C \in \mathbb{R}^{n\times n}$
such that $B = CAC^{-1}$.

$$[\det(C)][\det(C^{-1})] = \det(CC^{-1}) = \det(I) = 1$$

$$\begin{aligned}\det(B - \lambda I) &= \det(C(A - \lambda I)C^{-1}) \\ &= \cancel{[\det(C)]}[\det(A - \lambda I)]\cancel{[\det(C^{-1})]} \\ &= \det(A - \lambda I)\end{aligned}$$

$\det(A - \lambda I)$, $\det(B - \lambda I)$ have the same roots.
A, B have the same eigenvalues. QED

Fact: The eigenvalues of an upper triangluar matrix are its diagonal entries.

Proof (3×3 case):

$$\det\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} a-\lambda & b & c \\ 0 & d-\lambda & e \\ 0 & 0 & f-\lambda \end{bmatrix}$$

$$= (a-\lambda)(d-\lambda)(f-\lambda)$$

Eigenvalues: $a,\ d,\ f$ QED

Fact: Conjugation of a symmetric matrix by an orthogonal matrix yields a symmetric matrix.

Proof: Let $P \in \mathbb{R}^{n\times n}$ be symmetric.
Let $R \in \mathbb{R}^{n\times n}$ be orthogonal.

Want: $R^{-1}PR$ is symmetric.

$P^t = P$

$R^t = R^{-1}$

Want: $(R^{-1}PR)^t = R^{-1}PR$.

$$(R^{-1}PR)^t = (R^t PR)^t = R^t PR$$
$$= R^{-1}PR$$

QED

Fact: Conjugation of a symmetric matrix by an orthogonal matrix yields a symmetric matrix.

Definition:
A matrix M is **rotationally diagonalizable** if $\exists$rotation C s.t. $C^t MC$ is diagonal and real.

Fact: Any rotationally diagonalizable matrix is symmetric (and real).

The MOST IMPORTANT THEOREM in linear algebra

The Spectral Theorem:
Any symmetric (real) matrix is rotationally diagonalizable.

Proof later

The MOST IMPORTANT THEOREM in linear algebra

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The Spectral Theorem:
Any symmetric (real) matrix is rotationally diagonalizable.

The Spectral Corollary:
Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.
Then $\exists$ a rotation $R : \mathbb{R}^n \to \mathbb{R}^n$ s.t.
$Q \circ R$ is diagonal.

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Pf: $M := [Q]$ $\qquad Q = Q_M$

C^tMC is diagonal and real. Let $R := L_C$.

$Q \circ R = Q_M \circ L_C = Q_{C^tMC}$ is diagonal.

QED

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The idea: After chg. of var., any quad. form can be made diagonal.

The idea: Any quadr. form is equivalent to one that is easily studied.

The Spectral Corollary:

Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form.
Then $\exists$a rotation $R : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.
$Q \circ R$ is diagonal.

Application: Graph $x^2 + 4xy + 2y^2 = 2$.

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Application: Graph $x^2 + \boxed{4xy} + 2y^2 = 2$.

The idea: Graphing $ax^2 + by^2 = c$ is relatively easy.

The mixed term $4xy$ is the troublemaker term.

The spectral theorem says we can get rid of it, by rotating.

Specifically, . . .

The Spectral Corollary:

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Application: Graph $x^2 + 4xy + 2y^2 = 2$.

$$Q(x, y) := x^2 + 4xy + 2y^2$$

By the Spectral Corollary,
$\exists$a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$
s.t. $(Q \circ R)(x, y) = ax^2 + by^2$,
where $a = \dfrac{3 + \sqrt{17}}{2}$ and $b = \dfrac{3 - \sqrt{17}}{2}$
and where R can also be described explicitly.

Let $D := Q \circ R$, so $D(x, y) = ax^2 + by^2$.

$$Q(R(x,y)) = 2 \quad \text{iff} \quad D(x,y) = 2$$
$$R(x,y) \in \{Q = 2\} \text{ iff } (x,y) \in \{D = 2\}$$
$$\{Q = 2\} = R(\{D = 2\})$$

Graph $ax^2 + by^2 = 2$, then rotate by R.

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Let $D := Q \circ R$, so $D(x,y) = ax^2 + by^2$.

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Graph $ax^2 + by^2 = 2$, then rotate by R.

Application: Graph $x^2 + 4xy + 2y^2 = 2$.

Graph of $x^2 + 4xy + 2y^2 = 2$ is an hyperbola.

Graph of $ax^2 + by^2 = 2$ is an hyperbola.

$$a > 0 \qquad\qquad b < 0$$

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Let $D := Q \circ R$, so $D(x,y) = ax^2 + by^2$.

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and where R can also be described explicitly.

Next: How to find R, a and b?

$$Q(x, y) := x^2 + 4xy + 2y^2$$

Graph $ax^2 + by^2 = 2$, then rotate by R.

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$$Q(x,y) := x^2 + 4xy + 2y^2$$

We seek: a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ and $a, b \in \mathbb{R}$

s.t. $(Q \circ R)(x,y) = ax^2 + by^2$.

Application: Graph $x^2 + 4xy + 2y^2 = 2$.

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$$P := \begin{array}{cc} x & y \\ \left[\begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array}\right] & \begin{array}{c} x \\ y \end{array} \end{array} \qquad \begin{aligned} Q_P(x,y) &= x^2 + 4xy + 2y^2 \\ &= Q(x,y) \end{aligned}$$

$$\begin{aligned} \chi_P(\lambda) &= (1-\lambda)(2-\lambda) - 4 \\ &= \lambda^2 - 3\lambda - 2 \end{aligned}$$

eigenvalues: $a = \dfrac{3+\sqrt{17}}{2}$ and $b = \dfrac{3-\sqrt{17}}{2}$

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We seek: a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ and $a, b \in \mathbb{R}$ s.t. $(Q \circ R)(x,y) = ax^2 + by^2$.

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$$Q_P(x,y) = x^2 + 4xy + 2y^2 = Q(x,y)$$

$$\chi_P(\lambda) = (1-\lambda)(2-\lambda) - 4 = \lambda^2 - 3\lambda - 2$$

eigenvalues: $a = \dfrac{3+\sqrt{17}}{2}$ and $b = \dfrac{3-\sqrt{17}}{2}$

Fact: The eigenvalues of a symmetric (real) matrix are always real.

$$Q(x,y) := x^2 + 4xy + 2y^2$$

We seek: a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ and $a, b \in \mathbb{R}$

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eigenvalues: $a = \dfrac{3 + \sqrt{17}}{2}$ and $b = \dfrac{3 - \sqrt{17}}{2}$

Exercise: Find a rotation matrix R_0 s.t.

$L_{R_0}(1,0)$ is an a-eigenvector.

eigenvalues: $a =$ 2 2

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$$Q_P(x,y) = x^2 + 4xy + 2y^2 = Q(x,y)$$

eigenvalues: $a = \dfrac{3+\sqrt{17}}{2}$ and $b = \dfrac{3-\sqrt{17}}{2}$

Exercise: Find a rotation matrix R_0 s.t. $L_{R_0}(1,0)$ is an a-eigenvector.

$$PR_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = aR_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } R_0^{-1} P R_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Fact: Conjugation of a symmetric matrix by an orthogonal matrix yields a symmetric matrix.

$$R_0^{-1} P R_0 \in \begin{bmatrix} a & * \\ 0 & * \end{bmatrix} \qquad R_0^{-1} P R_0 \in \begin{bmatrix} a & 0 \\ 0 & \boxed{*} \end{bmatrix}$$

eigenvalues: a, b

$$P := \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \qquad \begin{aligned} Q_P(x,y) &= x^2 + 4xy + 2y^2 \\ &= Q(x,y) \end{aligned}$$

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eigenvalues: a, b

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Exercise: Find a rotation matrix R_0 s.t. $L_{R_0}(1,0)$ is an a-eigenvector.

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$$Q_P(x, y) = Q(x, y)$$

$$D_0 := R_0^{-1} P R_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$Q_P(x, y)$$

$$= Q(x, y)$$

$$a = \frac{3 + \sqrt{17}}{2} \text{ and } b = \frac{3 - \sqrt{17}}{2}$$

$$a = \frac{3 + \sqrt{17}}{2} \text{ and } b = \frac{3 - \sqrt{17}}{2}$$

$$Q_P(x,y) = Q(x,y)$$

$$R := L_{R_0}$$

$$\begin{aligned} Q \circ R &= Q_P \circ L_{R_0} \\ &= Q_{R_0^t P R_0} \\ &= Q_{D_0} \end{aligned}$$

$$D_0 := R_0^{-1} P R_0 = \begin{array}{ccc} x & y & \\ \left[\begin{matrix} a & 0 \\ 0 & b \end{matrix}\right. & \left.\right] & \begin{matrix} x \\ y \end{matrix} \end{array}$$

$$(Q \circ R)(x,y) = Q_{D_0}(x,y) = ax^2 + by^2$$

We seek: a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ and $a, b \in \mathbb{R}$ s.t. $(Q \circ R)(x,y) = ax^2 + by^2$.

Fact: A direct sum of rotationally diagonalizable matrices is rotationally diagonalizable.

Proof: Say $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$,

with A, B rotationally diagonalizable.

Want: C is rotationally diagonalizable.

Fix rotations X and Y s.t.

$X^{-1}AX$ and $Y^{-1}BY$ are diagonal.

Let $Z := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. Then Z is a rotation.

Then $Z^{-1}CZ = \begin{bmatrix} X^{-1}AX & 0 \\ 0 & Y^{-1}BY \end{bmatrix}$.

Then $Z^{-1}CZ$ is diagonal. QED

Fact: Any rotationally diagonalizable matrix
is symmetric (and real).

The Spectral Theorem:
Any symmetric (real) matrix is
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Let a be an eigenvalue of S. Then $a \in \mathbb{R}$.

Let $v \in \mathbb{R}^{3\times 1}$ be an a-eigenvector of S s.t. $|v| = 1$.

Let $e_1 \in \mathbb{R}^{3\times 1}$ have entries $1, 0, 0$.

Let R be a rotation matrix s.t. $Re_1 = v$.

$$SRe_1 = Sv = av = aRe_1$$

$$R^{-1}SRe_1 = ae_1$$

$$\boxed{R^{-1}SR} \in \begin{bmatrix} a & \boxed{*} & \boxed{*} \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

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diagonal

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$$\boxed{R^{-1}SR} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix} = [a] \oplus \begin{bmatrix} b & c \\ c & d \end{bmatrix}$$

symmetric

QED