Financial Mathematics

Vector fields and ordinary differential equations

Definition: A **vector field on** $\mathbb{R}^n$
is a smooth function $\mathbb{R}^n \to \mathbb{R}^n$.

(homo-
geneous)

Definition: A vector field V on $\mathbb{R}^n$
is **linear** if there is a matrix $M \in \mathbb{R}^{n \times n}$
such that, for all $p \in \mathbb{R}^n$,

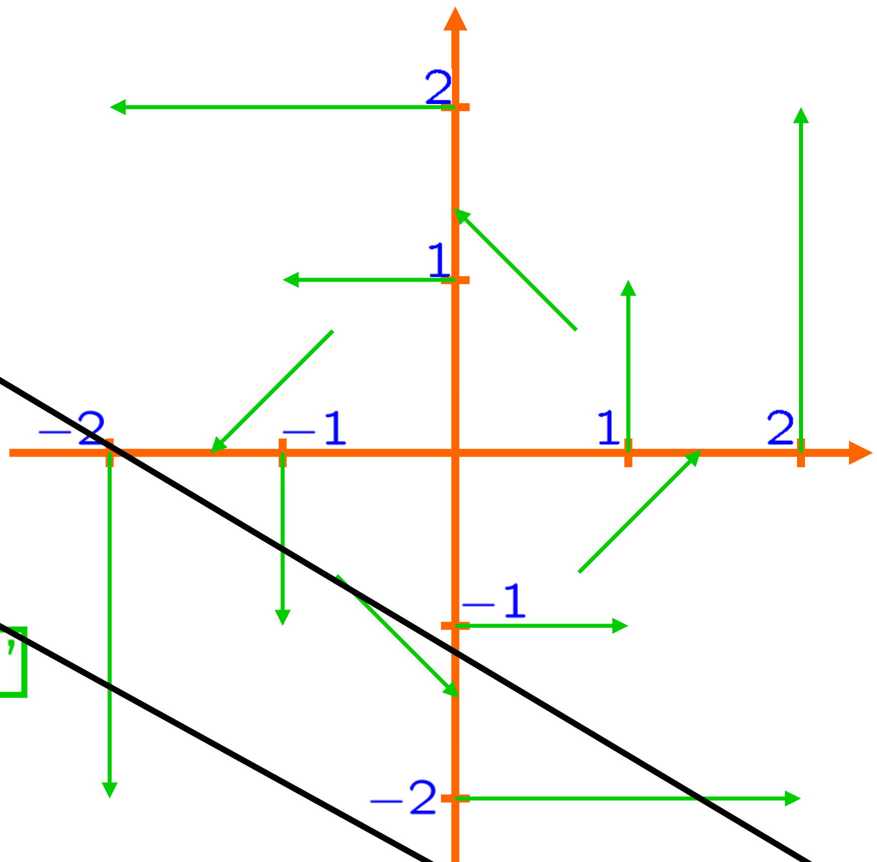
$$V(p) = \underbrace{L_M}_{\substack{\text{the linear map} \\ \text{corresponding to } M}}(p).$$

e.g.: If $M = \begin{bmatrix} -92 & 72 \\ -120 & 94 \end{bmatrix}$, $N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

then $L_M(x,y) = (\ -92x + 72y\ ,\ -120x + 94y\ )$,
and $L_N(x,y) = (-y, x) =: V(x,y)$.

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field.

Definition: A **flowline for** V is a smooth function $c : (a,b) \to \mathbb{R}^n$ such that
$\forall t \in (a,b)$, $\quad [c(t)]^\bullet = V(c(t))$,



$\boxed{V(x,y) = (-y,x)}$

“the tornado”

$(-y,x) =: V(x,y).$

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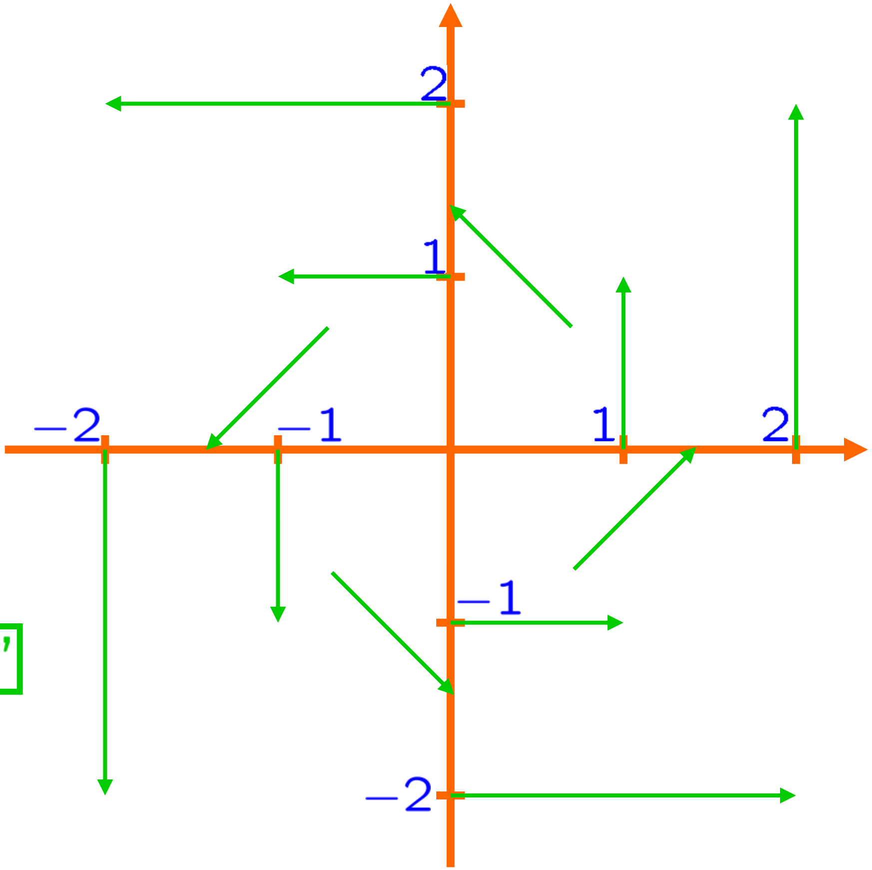

$V(x,y) = (-y,x)$

"the tornado"

Problem:

Find flowline c, s.t. $c(0) = (1,0)$.

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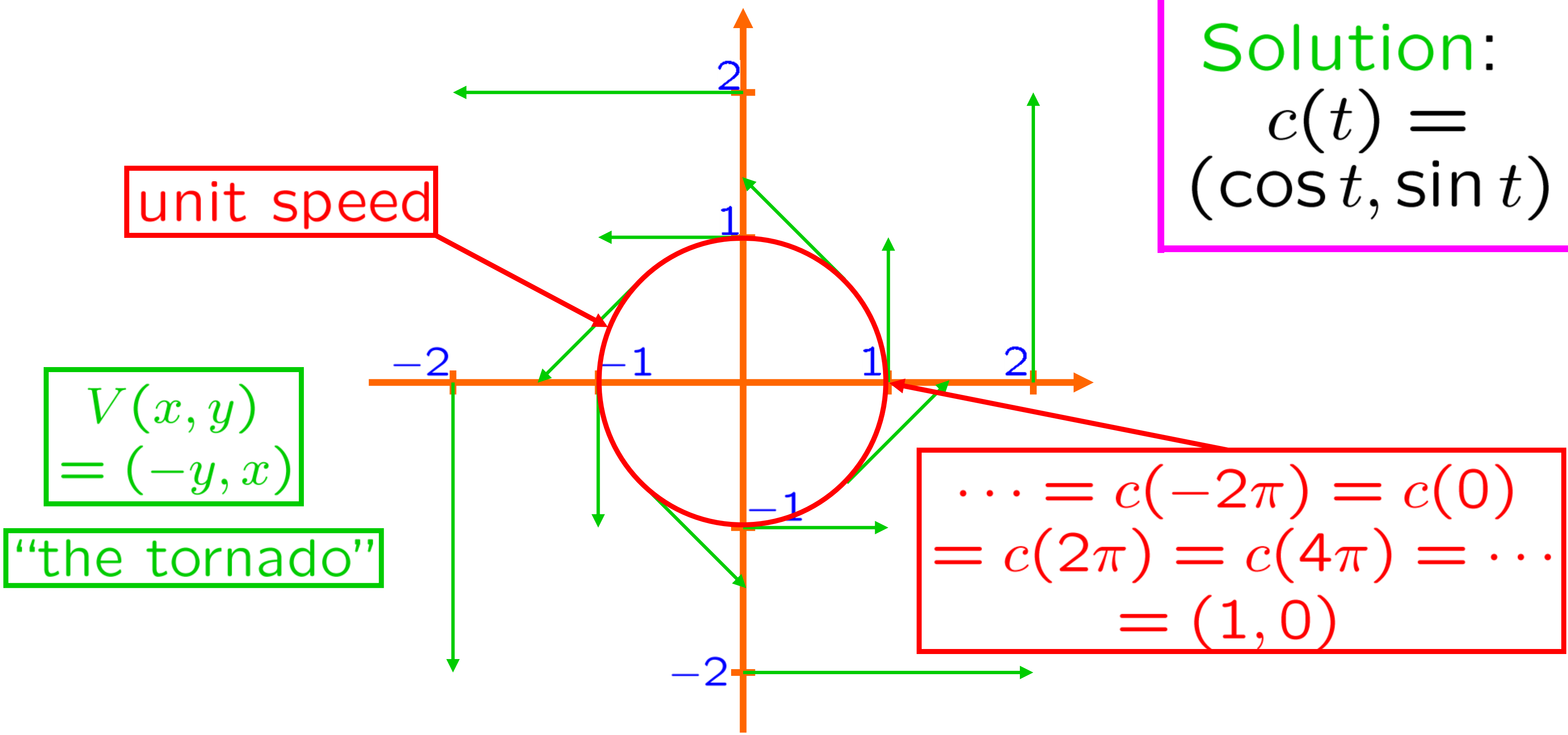

Solution: $c(t) = (\cos t, \sin t)$

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Solution: $c(t) = (\cos t, \sin t)$

☺

$$c(t) = (\cos t, \sin t)$$
$$[c(t)]^\bullet = (-\sin t, \cos t)$$
$$V(c(t)) = V(\cos t, \sin t)$$
$$= (-\sin t, \cos t) = [c(t)]^\bullet$$

$V(x,y) = (-y, x)$

"the tornado"

Problem:

Find flowline c, s.t. $c(0) = (1,0)$.

vector field

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a $\boxed{\text{VF}}$.

Question: For any $p \in \mathbb{R}^n$, does $\exists$a flowline $c : (a, b) \to \mathbb{R}^n$ for V s.t. $c(0) = p$?

Answer: Yes.

Question: For any $p \in \mathbb{R}^n$, does $\exists$a flowline $c : (-\infty, \infty) \to \mathbb{R}^n$ for V s.t. $c(0) = p$?

Answer: No:

Define $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ by $f(t) = \mathrm{Tan}(t)$.

Define $V : \mathbb{R} \to \mathbb{R}$ by $V(x) = f'(f^{-1}(x))$.

Then $V(f(t)) = f'(t)$; meaning? $x :\to f(t)$

f is a "maximal" flowline for V.

Definition: We say that a parametric curve $c:(a,b)\to\mathbb{R}^n$ is **footed at** p if:

$$0\in(a,b) \quad \text{and} \quad c(0)=p.$$

Definition: Let $c:(a,b)\to\mathbb{R}^n$, $c_0:(a_0,b_0)\to\mathbb{R}^n$ be flowlines. We say that c **extends** c_0 if

$$(a_0,b_0)\subseteq(a,b)$$

$$\text{and} \qquad \forall t\in(a_0,b_0), \quad c_0(t)=c(t).$$

Definition: Let V be a VF on $\mathbb{R}^n$. Let $p\in\mathbb{R}^n$. A flowline for V footed at p is said to be **maximal** if it extends any other flowline for V footed at p.

Fact: $\forall$VF V on $\mathbb{R}^n$, $\forall p\in\mathbb{R}^n$, $\exists!$max. flowline for V footed at p.

Define $f:(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$ by $f(t)=\mathrm{Tan}(t)$.

Define $V:\mathbb{R}\to\mathbb{R}$ by $V(x)=f'(f^{-1}(x))$.

Then $V(f(t))=f'(t)$; Why?ing?

f is a "maximal" flowline for V.

Fact: $\forall$VF V on $\mathbb{R}^n$, $\forall p \in \mathbb{R}^n$,
$\exists!$max. flowline for V footed at p.

Fact: The maximal flowline footed at p
extends all flowlines footed at p.

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$\exists$!max. flowline for V footed at p.

Fact: The maximal flowline footed at p
extends all flowlines footed at p.

$$\lim_{t \to \pi/2^+} (f(t)) = -\infty \quad \text{and} \quad \lim_{t \to \pi/2^-} (f(t)) = \infty$$

No flowline extends f.

f extends every flowline. $\nexists$c

Question: For any $p \in \mathbb{R}^n$, does $\exists$a flowline
$c : (-\infty, \infty) \to \mathbb{R}^n$ for V s.t. $c(0) = p$?

Answer: No:

Define $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ by $f(t) = \text{Tan}(t)$.

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Ordinary Differential Equations and Vector Fields

ODEs

Problem: Find an expression X of t such that

$$\boxed{\frac{dX}{dt} = (0.05)X}, \qquad \boxed{[X]_{t=0} = 1}.$$

ODE

Initial value condition

Problem: Let V be the vector field on $\mathbb{R}$ defined by $V(x) = (0.05)x$.
Find a flowline c for V footed at 1.

Let $X := c(t)$.
Then $dX/dt = c'(t) = V(c(t))$
$= [0.05][c(t)] = (0.05)X$
and $[X]_{t=0} = c(0) = 1$.

Ordinary Differential Equations and Vector Fields

Problem: Find expressions X, Y of t such that

System of ODEs

$$\boxed{\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,}$$

Initial value conditions

$$\boxed{[X]_{t=0} = 3, \qquad [Y]_{t=0} = 8}.$$

Problem: Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) = (-x + y, 2x - 4y)$.
Find a flowline c for V footed at $(3, 8)$.

Ordinary Differential Equations and Difference Equations

The idea: To get flowline of V footed at p:

For each integer $N > 0$:

$$c_N(0) := p.$$

$$c_N(1/N) := [c_N(0)] + [1/N][V(c_N(0))].$$

$$c_N(2/N) := [c_N(1/N)] + [1/N][V(c_N(1/N))].$$

$$c_N(3/N) := [c_N(2/N)] + [1/N][V(c_N(2/N))].$$

$$\vdots$$

Similarly, define $c_N(-1/N), c_N(-2/N), \ldots$.

Piecewise linear interpolation: $c_N : \mathbb{R} \to \mathbb{R}$.

Let $c := \lim_{N \to \infty} c_N$.

Then c is a flowline of V.

"the Euler method"

Let $V:\mathbb{R}^n \to \mathbb{R}^n$ be a vector field.

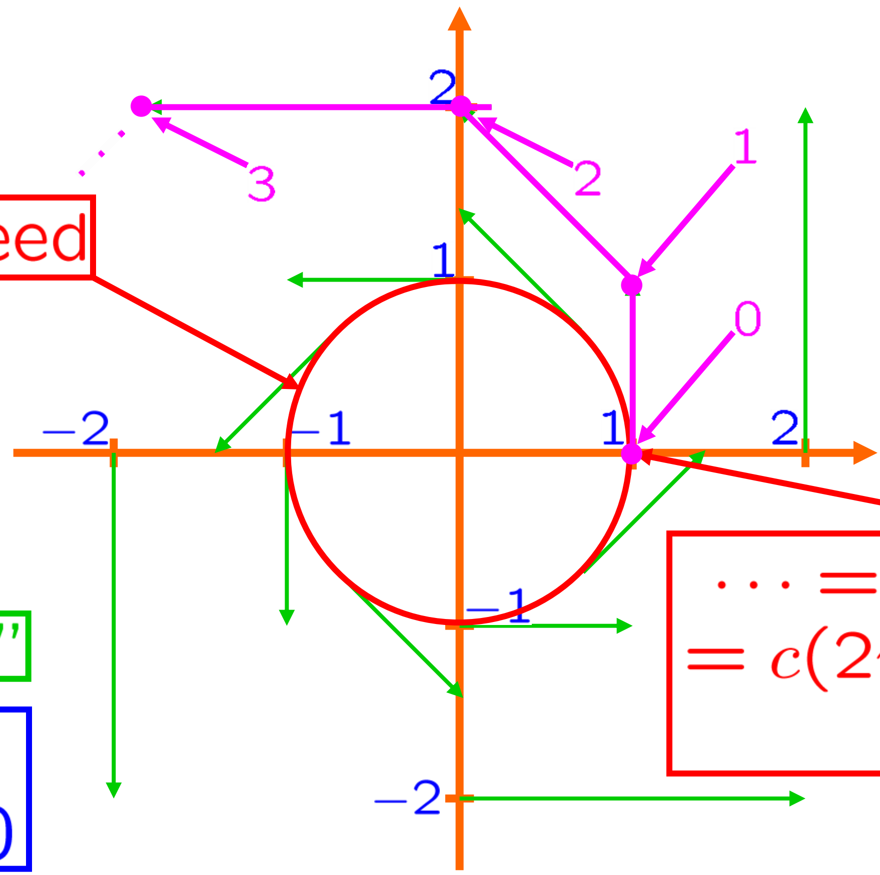
Definition: A **flowline for** V is a smooth function $c:(a,b)\to\mathbb{R}^n$ such that

$\forall t \in (a,b)$, $\qquad [c(t)]^\bullet = V(c(t))$,

$N = 1$

unit speed

Solution: $c(t) = (\cos t, \sin t)$



$V(x,y) = (-y,x)$

"the tornado"

$\cdots = c(-2\pi) = c(0) = c(2\pi) = c(4\pi) = \cdots = (1,0)$

Exercise: $N = 10$

Problem:

Find flowline c, s.t. $c(0) = (1,0)$.

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute $[X]_{t=2}$.

Problem: Find an expression X of t
defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$
by the formulas

$$\boxed{[X]_{t\to t+\triangle t} - X} \rightarrow \triangle X, \qquad \boxed{0.1} \rightarrow \triangle t$$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$.

Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

$\boxed{[X]_{t \to t+\triangle t} - X}$ $\boxed{0.1}$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$.

Let $A(t) := X$.

Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

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$\boxed{0.1}$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$.

Let $A(t) := X$. $\qquad A(0) := [X]_{t=0} = 1$.

$$\frac{[A(t+0.1)] - [A(t)]}{0.1} = (0.05)[A(t)]$$

$$\begin{aligned}[A(t+0.1)] - [A(t)] &= (0.1)(0.05)[A(t)] \\ &= (1/10)(0.05)[A(t)]\end{aligned}$$

$$A(t+0.1) = [1 + (1/10)(0.05)][A(t)]$$

Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

$$\boxed{[X]_{t\to t+\triangle t} - X} \rightarrow \triangle X \qquad \boxed{0.1} \rightarrow \triangle t$$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$.

Let $A(t) := X$. $\qquad A(0) := [X]_{t=0} = 1.$

$$A(t + 0.1) = [1 + (1/10)(0.05)][A(t)]$$

$$[X]_{t=2} = A(2) = [1 + (1/10)(0.05)][A(1.9)]$$

$$A(t + 0.1) = [1 + (1/10)(0.05)][A(t)]$$

Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

$$\frac{\Delta X}{\Delta t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

$\boxed{[X]_{t \to t+\Delta t} - X} \to \Delta X$, $\quad \boxed{0.1} \to \Delta t$

Find $[X]_{t=2}$. **Solution:** $[1 + (1/10)(0.05)]^{20}$

Let $A(t) := X$. $\qquad A(0) := [X]_{t=0} = 1.$

$$A(t + 0.1) = [1 + (1/10)(0.05)][A(t)]$$

Start with \$1 in a bank account at time 0.
Every 1/10 years, add on 1/10 of 5% interest.
How much in account 2 years later?

$$\begin{aligned}[X]_{t=2} = A(2) &= [1 + (1/10)(0.05)][A(1.9)] \\ &= [1 + (1/10)(0.05)]^2[A(1.8)] \\ &= \cdots = [1 + (1/10)(0.05)]^{20}[\cancel{A(0)}]\end{aligned}$$

Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

$\boxed{[X]_{t\to t+\triangle t} - X} \to \triangle X$

$\boxed{0.1} \to \triangle t$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$. **Solution:** $[1+(1/10)(0.05)]^{20}$

Compounding 10 times per year, with nominal annual interest rate: 5%

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Problem: Find an expression X of t defined at $t = \ldots, -0.2, -0.1, 0, 0.1, 0.2, \ldots$ by the formulas

$$\frac{\Delta X}{\Delta t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

$$\Delta X = [X]_{t \to t+\Delta t} - X, \qquad \Delta t = 0.1$$

Find $[X]_{t=2}$. Solution: $[1 + (1/10)(0.05)]^{20}$

Compounding 10 times per year, with nominal annual interest rate: 5%

Problem: Find an expression X of t defined at $t = \ldots, -0.02, -0.01, 0, 0.01, 0.02, \ldots$ such that

$$\frac{\Delta X}{\Delta t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

$$\Delta X = [X]_{t \to t+\Delta t} - X, \qquad \Delta t = 0.01$$

Find $[X]_{t=2}$.

Compounding 100 times per year, with nominal annual interest rate: 5%

Solution: $[1 + (1/100)(0.05)]^{200}$

Problem: Find an expression X of t defined at $t = \ldots, -2/N, -1/N, 0, 1/N, 2/N, \ldots$ by the formulas

$$\frac{\Delta X}{\Delta t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

ΔX: $[X]_{t \to t+\Delta t} - X$

Δt: $1/N$

Find $[X]_{t=2}$.

Compounding N times per year, with nominal annual interest rate: 5%

Solution: $[1 + (1/N)(0.05)]^{2N}$

Problem: Find an expression X of t defined at $t = \ldots, -0.02, -0.01, 0, 0.01, 0.02, \ldots$ such that

$$\frac{\Delta X}{\Delta t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

ΔX: $[X]_{t \to t+\Delta t} - X$

Δt: 0.01

Find $[X]_{t=2}$.

Compounding 100 times per year, with nominal annual interest rate: 5%

Solution: $[1 + (1/100)(0.05)]^{200}$

Problem: Find an expression X of t defined at $t = \ldots, -2/N, -1/N, 0, 1/N, 2/N, \ldots$ by the formulas

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$$\boxed{1/N} \to \triangle t$$

$$\frac{\triangle X}{\triangle t} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Find $[X]_{t=2}$.

Compounding N times per year, with nominal annual interest rate: 5%

Solution: $[1 + (1/N)(0.05)]^{2N}$

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute $[X]_{t=2}$.

Solution: $\lim_{N\to\infty} [1 + (1/N)(0.05)]^{2N} = e^{(0.05)2}$

Continuous compounding, with nominal annual interest rate: 5%

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute $\boxed{X}$.

Solution: $\lim_{N\to\infty} [1 + (1/N)(0.05)]^{\boxed{t}N} = e^{(0.05)\boxed{t}}$

Continuous compounding, with nominal annual interest rate: 5%

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute $\boxed{[X]_{t=2}}$.

Solution: $\lim_{N\to\infty} [1 + (1/N)(0.05)]^{\boxed{2}N} = e^{(0.05)\boxed{2}}$

Continuous compounding, with nominal annual interest rate: 5%

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$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute X. Solution: $X = e^{(0.05)t}$

Solution: $= e^{(0.05)t}$

Check: $[X]_{t=0} = [e^{(0.05)t}]_{t=0} = 1$

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute X. Solution: $X = e^{(0.05)t}$

Check: $[X]_{t=0} = [e^{(0.05)t}]_{t=0} = 1$

$$\frac{dX}{dt} = \frac{d}{dt}[e^{(0.05)t}] = [e^{(0.05)t}][0.05] = (0.05)X$$

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute X. Solution: $X = e^{(0.05)t}$

Alternate solution:

$Y := \ln X$ CHAIN RULE!

$$\frac{dY}{dt} = \frac{1}{X}\frac{dX}{dt} = \frac{1}{\cancel{X}}(0.05)\cancel{X} = 0.05$$

$$[Y]_{t=0} = [\ln X]_{t=0} = \ln 1 = 0$$

$$Y = (0.05)t$$

$$X = e^{\ln X} = e^{Y} = e^{(0.05)t} \blacksquare$$

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute X. Solution: $X = e^{(0.05)t}$

Third approach:

$$\frac{d}{dt}(c(t)) = (0.05)(c(t)), \qquad c(0) = 1.$$

is equivalent to

$$c(t) = \boxed{1 + \int_0^t (0.05)(c(s))\, ds}$$

is equivalent to

$\boxed{\mathcal{A}(c) = c}$

$\mathcal{F} := \{\text{contin. fns } [-10, 10] \to \mathbb{R}\}$.

Define $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ by the rule:

c is a fixpoint of $\mathcal{A}$

$\forall h \in \mathcal{F}$, the function $\mathcal{A}(h) \in \mathcal{F}$ is defined by:

$c(t) = (\mathcal{A}(c))(t)$

$$(\mathcal{A}(\underset{c}{h}))(t) = \boxed{1 + \int_0^t (0.05)(\underset{c}{h}(s))\, ds}.$$

Let $\mathcal{F} := \{\text{contin. fns } [-10,10] \to \mathbb{R}\}$.
Define $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ by the rule:

$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds.$$

Fact: There is a distance (a "metric") in $\mathcal{F}$

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$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds.$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\text{dist}(g, h)].$$

$\mathcal{A}$ is a contraction w.r.t. this distance.

Contraction factor is 0.5

Let $\mathcal{F} := \{$contin. fns $[-10, 10] \to \mathbb{R}\}$.
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Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}$, $\quad \mathrm{dist}(p, q) := \max\limits_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \mathrm{dist}(g, h)$.
Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \mathrm{dist}(g_0, h_0)$.
Want: $d_0 \le [0.5]d$

$$\forall \overset{s}{t} \in I, |(g(\overset{s}{t})) - (h(\overset{s}{t}))| \le d$$

Let $\mathcal{F}$:= {contin. fns $[-10, 10] \to \mathbb{R}$}.
Define $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ by the rule:

$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds.$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

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Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$$\overset{g_0}{h_0}(t) = (\mathcal{A}(\overset{g}{h}))(t) = 1 + \int_0^t [0.05][\overset{g}{h}(s)]\,ds$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

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Fix $g, h \in \mathcal{F}$, and let $d := \mathrm{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \mathrm{dist}(g_0, h_0)$.

Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$$(g_0 - h_0)(t) = \cancel{(1-1)} + \int_0^t [0.05][(g(s)) - (h(s))]\, ds$$

$$\overset{g_0}{h_0}(t) = (\mathcal{A}(\overset{g}{h}))(t) = 1 + \int_0^t [0.05][\overset{g}{h}(s)]\, ds$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \le [0.5][\text{dist}(g, h)].$$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}, \quad \text{dist}(p, q) := \max\limits_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \text{dist}(g_0, h_0)$.

Want: $d_0 \le [0.5]d$

$\boxed{\forall s \in I, |(g(s)) - (h(s))| \le d}$

$$(g_0 - h_0)(t) = \cancel{(1-1)} + \int_0^t [0.05][(g(s)) - (h(s))]\,ds$$

$$(g_0(t)) - (h_0(t)) = \int_0^t [0.05][(g(s)) - (h(s))]\,ds$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\text{dist}(g,h)].$$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}$, $\boxed{\text{dist}(\underset{g_0}{p}, \underset{h_0}{q}) := \max_{t \in I} |(\underset{g_0}{p}(t)) - \underset{h_0}{q}(t))|}$

Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g,h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \text{dist}(g_0, h_0)$.

Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$$\text{dist}(g_0, h_0) := \boxed{\max_{t \in I}} |(g_0(t)) - h_0(t))|$$

$$|(g_0(t)) - (h_0(t))| = \left| \int_0^t [0.05][(g(s)) - (h(s))] \, ds \right|$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\text{dist}(g, h)].$$

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Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g, h)$.

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$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$$\mathrm{dist}(g_0, h_0) := \max_{t \in I} |(g_0(t)) - h_0(t))|$$

$$= \max_{t \in I} \left| \int_0^t [0.05][(g(s)) - (h(s))] \, ds \right|$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\mathrm{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\mathrm{dist}(g, h)].$$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}$, $\quad \mathrm{dist}(p, q) := \max_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \mathrm{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \mathrm{dist}(g_0, h_0)$.

Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$\text{dist}(g_0, h_0)$

$$\text{dist}(g_0, h_0) = \max_{t \in I} \left| \int_0^t [0.05][(g(s)) - (h(s))]\, ds \right|$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \le [0.5][\text{dist}(g, h)].$$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}, \quad \text{dist}(p, q) := \max_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \text{dist}(g_0, h_0)$.

Want: $d_0 \le [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \le d$

$$d_0 = \max_{t \in I} \left| \int_0^t [0.05][(g(s)) - (h(s))]\, ds \right|$$

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\text{dist}(g, h)].$$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}, \quad \text{dist}(p, q) := \max_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \text{dist}(g_0, h_0)$.

Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

$$d_0 = \max_{t \in I} \left| \int_0^t [0.05] \underbrace{[(g(s)) - (h(s))]}_{\text{between } \pm d} \, ds \right|$$

$$\underbrace{}$$

between $\pm t[0.05]d$

so between $\pm \boxed{10[0.05]}d$

so between $\pm \boxed{[0.5]}d$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}, \quad \mathrm{dist}(p, q) := \max_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \mathrm{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \mathrm{dist}(g_0, h_0)$.

Want: $d_0 \leq [0.5]d$

$\forall s \in I, |(g(s)) - (h(s))| \leq d$

QED

$$d_0 \leq [0.5]d = \max_{t \in I} \overbrace{\left| \int_0^t [0.05][\underbrace{(g(s)) - (h(s))}_{\text{between } \pm d}]\, ds \right|}^{\text{between } 0 \text{ and } [0.5]d}$$

between $\pm t[0.05]d$

so between $\pm 10[0.05]d$

so between $\pm[0.5]d$

Proof: $I := [-10, 10]$

$\forall p, q \in \mathcal{F}, \quad \text{dist}(p, q) := \max_{t \in I} |(p(t)) - q(t))|$

Fix $g, h \in \mathcal{F}$, and let $d := \text{dist}(g, h)$.

Let $g_0 := \mathcal{A}(g)$, $h_0 := \mathcal{A}(h)$, $d_0 := \text{dist}(g_0, h_0)$.

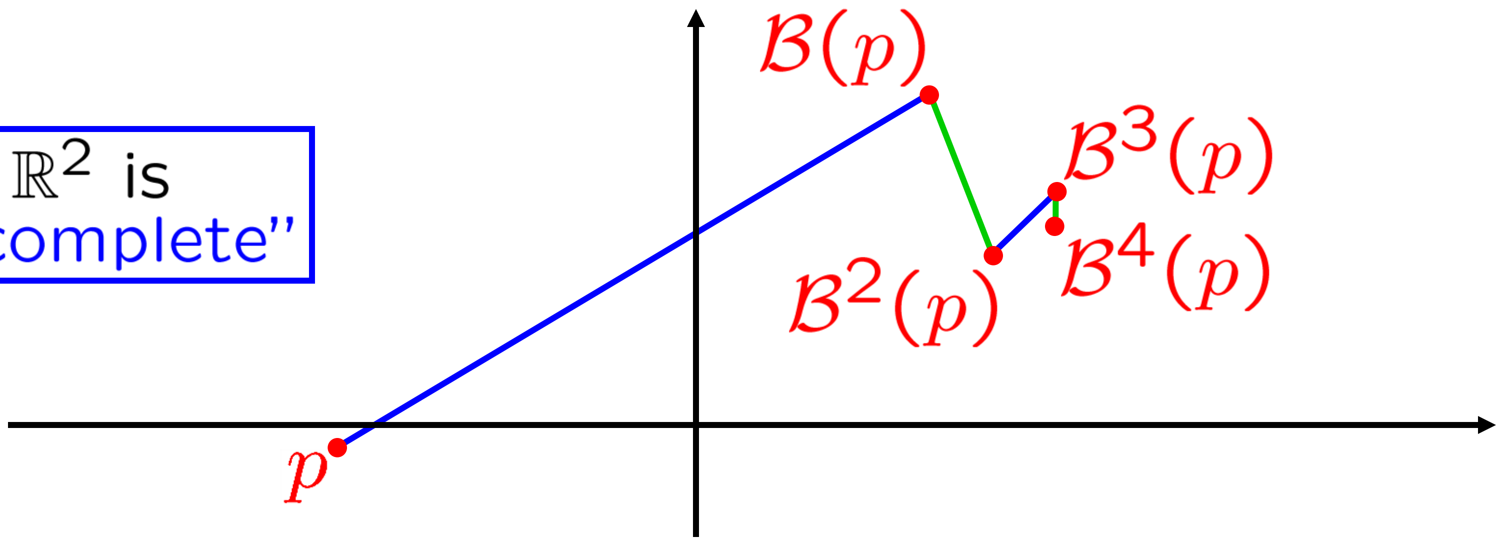
Want: $d_0 \leq [0.5]d$

$\boxed{\forall s \in I, |(g(s)) - (h(s))| \leq d}$

Suppose $\mathcal{B} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, and
$\forall q, r \in \mathbb{R}^2, \quad \text{dist}(\mathcal{B}(q), \mathcal{B}(r)) \leq [0.5][\text{dist}(q, r)]$.

The (forward) orbit of p

$\mathbb{R}^2$ is "complete"



$$\boxed{p,} \quad \mathcal{B}(p),\ \mathcal{B}^2(p),\ \mathcal{B}^3(p),\ \mathcal{B}^4(p),\ \ldots \ \to \ q$$
$$\|$$
$$\mathcal{B}(p),\ \mathcal{B}^2(p),\ \mathcal{B}^3(p),\ \mathcal{B}^4(p),\ \mathcal{B}^5(p),\ \ldots \ \to \ \mathcal{B}(q)$$

Key point: For a contraction mapping, the lim of any forward orbit is a fixpt.

Let $\mathcal{F} := \{\text{contin. fns } [-10,10] \to \mathbb{R}\}$.
Define $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ by the rule:

$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds.$$

---

complete

Fact: There is a distance (a "metric") in $\mathcal{F}$ such that, $\forall g, h \in \mathcal{F}$,

$$\text{dist}(\mathcal{A}(g), \mathcal{A}(h)) \leq [0.5][\text{dist}(g,h)].$$

$\mathcal{A}$ is a contraction w.r.t. this distance.

with respect to

Fact: $\forall f \in \mathcal{F}$,
the sequence $\overbrace{f, \mathcal{A}(f), \mathcal{A}^2(f), \mathcal{A}^3(f), \ldots}^{\text{the forward orbit of } f \text{ under } \mathcal{A}}$
converges in $\mathcal{F}$ to a fixpoint of $\mathcal{A}$.

Problem: Find an expression X of t such that

$$\frac{dX}{dt} = (0.05)X, \qquad [X]_{t=0} = 1.$$

Compute X. Solution: $X = e^{(0.05)t}$

Third approach:

$$\frac{d}{dt}(c(t)) = (0.05)(c(t)), \qquad c(0) = 1.$$

is equivalent to $\mathcal{A}(c) = c$.

$\mathcal{F} = \{\text{contin. fns } [-10, 10] \to \mathbb{R}\} \qquad \mathcal{A} : \mathcal{F} \to \mathcal{F}$

$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds$$

Define $f \in \mathcal{F}$ by $f(t) = 1$.
Let c be the limit of $f, \mathcal{A}(f), \mathcal{A}^2(f), \ldots$.

$$(\mathcal{A}(f))(t) = 1 + (0.05)t$$

$$(\mathcal{A}^2(f))(t) = 1 + (0.05)t + (0.05)^2(t^2/(2!))$$

$$(\mathcal{A}^3(f))(t) = 1 + (0.05)t + (0.05)^2(t^2/(2!)) + (0.05)^3(t^3/(3!))$$

$$\vdots \qquad\qquad \vdots$$

$$c(t) = e^{(0.05)t}$$

Note: Works for all $t \in \mathbb{R}$, not just $t \in [-10, 10]$

Sol'n: $c(t) = e^{(0.05)t}$

$\mathcal{F} = \{$contin. fns $[-10, 10] \to \mathbb{R}\}$ $\qquad \mathcal{A} : \mathcal{F} \to \mathcal{F}$

$$(\mathcal{A}(h))(t) = 1 + \int_0^t [0.05][h(s)]\, ds$$

Define $f \in \mathcal{F}$ by $f(t) = 1$.

Let c be the limit of $f, \mathcal{A}(f), \mathcal{A}^2(f), \ldots$.

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

$$[X]_{t=0} = 3, \qquad [Y]_{t=0} = 8.$$

Problem 2: Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) = (-x + y, 2x - 4y)$. Find a flowline c for V footed at $(3, 8)$.

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

$$Z := \begin{bmatrix} X \\ Y \end{bmatrix} \qquad A := \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$$

$$\dot{Z} = \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} -X + Y \\ 2X - 4Y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\dot{Z} = AZ$$

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

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$$\dot{Z} = AZ$$

$$\frac{d}{dt} e^{tA} = A e^{tA}$$

$$\dot{Z} = AZ$$

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

$$\underset{2 \times 1}{Z} := \begin{bmatrix} X \\ Y \end{bmatrix} \qquad A := \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$$

$$\dot{Z} = AZ$$

$$2 \times 2 \qquad \frac{d}{dt} e^{tA} = A e^{tA} \qquad \cancel{Z = e^{tA}}$$

$$\forall C \in \mathbb{R}^{2 \times 1}, \quad \frac{d}{dt}[e^{tA} C] = A e^{tA} C$$

$$Z = e^{tA} C$$

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

Problem: $\dot{B} = (0.05)B$

Solutions: $\forall C \in \mathbb{R}$,

$B = Ce^{(0.05)t}$ is a solution.

Problem: $\dot{B} = aB$

Solutions: $\forall C \in \mathbb{R}$,

$B = Ce^{at}$ is a solution.

Problem: $\dot{Z} = AZ$

Solutions: $\forall C \in \mathbb{R}^{2\times 1}$,

$Z = e^{tA}C$ is a solution.

$\forall C \in \mathbb{R}^{2\times 1}$, $\quad \frac{d}{dt}[e^{tA}]C = Ae^{tA}C$

$Z = e^{tA}C$

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

$$[X]_{t=0} = 3, \qquad [Y]_{t=0} = 8.$$

$$Z := \begin{bmatrix} X \\ Y \end{bmatrix} \qquad A := \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$$

$$\dot{Z} = AZ$$

$\forall C \in \mathbb{R}^{2\times 1}$,

$Z = e^{tA}C$ is a solution.

$$\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}_{t=0} = [Z]_{t=0} = [e^{tA}C]_{t=0} = IC = C$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = Z = e^{tA}C = (\exp(tA))C$$

$$= \left(\exp\left(t\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}\right)\right)\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$Z := \begin{bmatrix} X \\ Y \end{bmatrix} \qquad A := \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$$

$$\dot{Z} = AZ$$

$\forall C \in \mathbb{R}^{2\times 1}$,

$Z = e^{tA}C$ is a solution.

$$\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}_{t=0} = [Z]_{t=0} = [e^{tA}C]_{t=0} = IC = C$$

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$$= \left(\exp\left(t\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}\right)\right)\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \left(\exp\left(t\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}\right)\right)\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Problem 1: Find expressions X, Y of t such that

$$\frac{dX}{dt} = -X + Y, \qquad \frac{dY}{dt} = 2X - 4Y,$$

$$[X]_{t=0} = 3, \qquad [Y]_{t=0} = 8.$$

Solution: $$\begin{bmatrix} X \\ Y \end{bmatrix} = \left(\exp \left(t \underbrace{\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}}_{A} \right) \right) \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Exercises: Find an invertible $B \in \mathbb{R}^{2\times 2}$ s.t. $D := BAB^{-1}$ is diagonal

Compute $\exp(tD)$.

Compute $B^{-1}(\exp(tD))B = \exp(tA)$.

SKILLS: Integrate a constant vector field.
Integrate a homogeneous linear vector field.
Integrate an inhomogeneous linear vector field.

Solution: $$\begin{bmatrix} X \\ Y \end{bmatrix} = \left(\exp \left(t \underbrace{\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}}_{A} \right) \right) \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Exercises: Find an invertible $B \in \mathbb{R}^{2\times 2}$
s.t. $D := BAB^{-1}$ is diagonal

Compute $\exp(tD)$.

Compute $B^{-1}(\exp(tD))B = \exp(tA)$.

SKILLS: Integrate a constant vector field.
Integrate a homogeneous linear vector field.
Integrate an inhomogeneous linear vector field.

Example: Find the flowline, footed at $(8,9)$,
for $V(x,y) = (3x + 2y + 5\,,\ 2x + 3y + 7\ )$.
inhomogeneous linear

Corresponding homogeneous linear:
Example: Find the flowline, footed at $(8,9,1)$,
for $W(x,y,z) = (3x + 2y + 5z, 2x + 3y + 7z, 0)$.

Last coordinate is always 1.

First two coordinates solve
the inhomgeneous problem.

Next: 2nd deriv test

The second derivative test

Terminology:

A function taking values in the scalars (*i.e.*, in the real numbers) is often called a **functional**.

So **functional** = scalar-valued function.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth functional.

Goal: Maximize and/or minimize f.

Definition: f attains a **local max** at $a \in \mathbb{R}^n$ if $\exists \delta > 0$ s.t.: $|x - a| < \delta$ implies $f(x) \leq f(a)$.

Definition: f attains a **local min** at $a \in \mathbb{R}^n$ if $\exists \delta > 0$ s.t.: $|x - a| < \delta$ implies $f(x) \geq f(a)$.

The second derivative test

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth functional.

Goal: Maximize and/or minimize f.

Definition: A **critical point for** f is a point $a \in \mathbb{R}^n$ s.t. $f'(a) = 0$.

Fact: If f attains a local max or local min at a, then a is a critical point for f.

Second derivative test:

Let $[C(x)] + [L(x)] + [(Q(x))/(2!)]$ be the second order Macl. approx. of $f(x + a)$.

Assume that a is a critical point of f, *i.e.*, that $L = 0$.

If Q is positive definite, then f attains a local min at a.

neg. def. $\Rightarrow$ loc. max.

$C = C_{f(a)}$

$L = L_{f'(a)}$

$Q = Q_{f''(a)}$

The gradient and reverse-gradient flows

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth functional.

Goal: Minimize f.

Approach:

Pick a "starting point" $s \in \mathbb{R}^n$.

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be the **reverse gradient VF**, defined by $V(x) = -(\nabla f)(x)$.

Let c be a flowline for V footed at s.

Hope that $c(t)$ is defined for all $t \geq 0$.

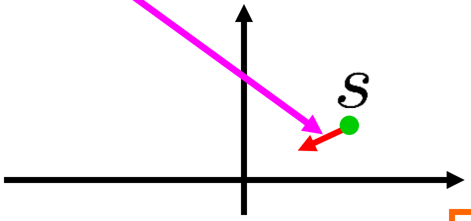
Hope that $a := \lim_{t\to\infty} c(t)$ exists.

critical point

Hope that f attains its minimum at a.

e.g.: $f(x,y) = (x^2 + y^2)/4$

$-(\nabla f)(x,y) = (-x/2, -y/2)$



$$c(t) = e^{-t/2}s \to 0, \text{ as } t \to \infty$$

The gradient and reverse-gradient flows

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth functional.

Goal: Maximize f.

Approach:

Pick a "starting point" $s \in \mathbb{R}^n$.

Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be the **gradient VF**,
defined by $V(x) = (\nabla f)(x)$.

Let c be a flowline for V footed at s.

Hope that $c(t)$ is defined for all $t \geq 0$.

Hope that $a := \lim_{t \to \infty} c(t)$ exists.

critical point

Hope that f attains its maximum at a.

STOP