Financial Mathematics

Multivariable change of variables

START OF MULTIVARIABLE INTEGRAL CALCULUS

Single variable change of variables formula

Theorem:

Let $D, E \subseteq \mathbb{R}$ be open.
Assume $\psi : D \to E$ is smooth and bijective.
Assume $f : E \to \mathbb{R}$ is continuous.

Then $$\int_E f(x)\,dx = \int_D [f(\psi(s))]\,[|\psi'(s)|]\,ds$$

Special case:

$D = (a, b)$, ψ increasing, $E = (\psi(a), \psi(b))$

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b [f(\psi(s))][\psi'(s)]\,ds$$

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Then $$\int_E f(x)\,dx = \int_D [f(\psi(s))]\,[|\psi'(s)|]\,ds$$

Special case:

$D = (a, b)$, ψ decreasing, $E = (\psi(b), \psi(a))$

$$\int_{\psi(b)}^{\psi(a)} f(x)\,dx = \int_a^b [f(\psi(s))][-\psi'(s)]\,ds$$

Single variable change of variables formula

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Special case:

$D = (a, b)$, ψ decreasing, $E = (\psi(b), \psi(a))$

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b [f(\psi(s))][\psi'(s)]\,ds$$

Note: This last formula is true even if ψ is not bijective.

Single variable change of variables formula

Theorem:

Let $D, E \subseteq \mathbb{R}$ be open.

Assume $\psi : D \longrightarrow E$ is smooth and bijective.

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Then $$\int_E f(x)\,dx = \int_D [f(\psi(s))]\,[|\psi'(s)|]\,ds$$

Goal: Find a similar formula of the form

$$\int\int_E f(x,y)\,dx\,dy = \int\int_D [f(\psi(s,t))][?????]\,ds\,dt,$$

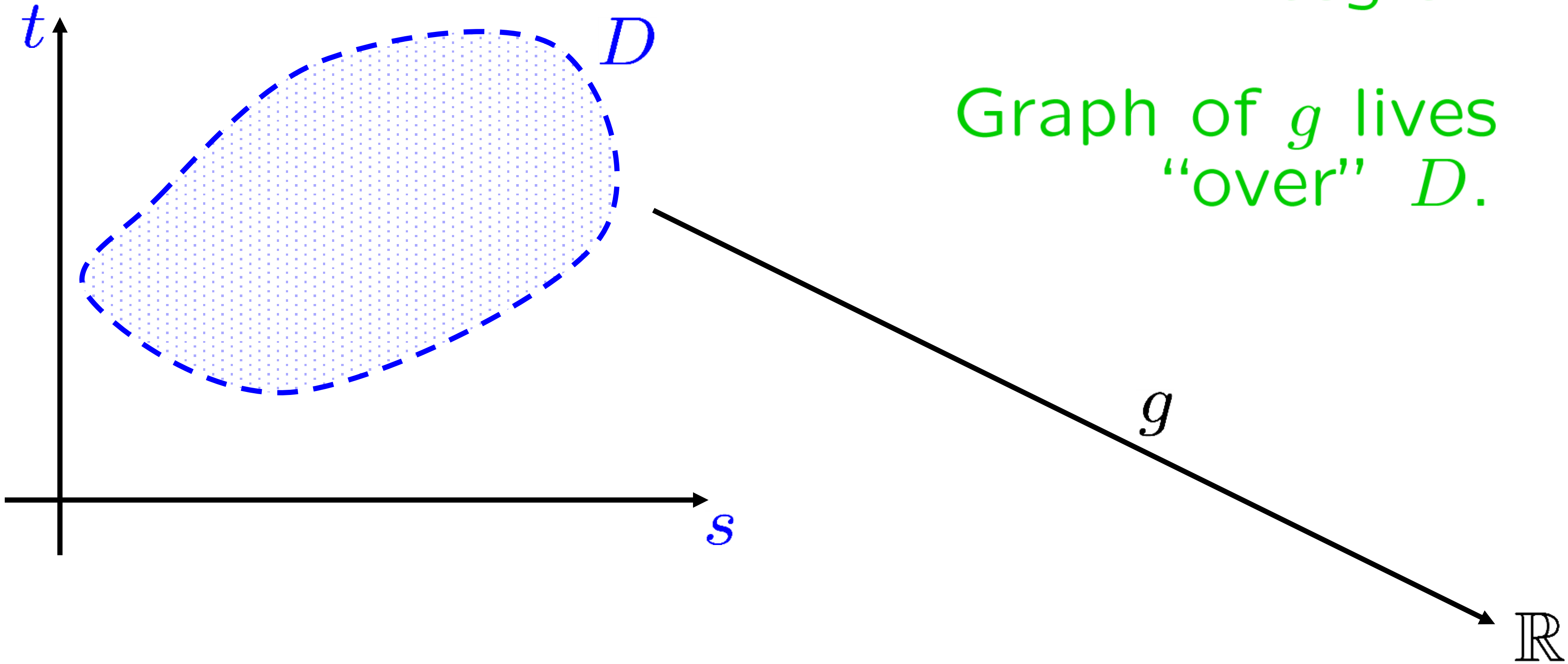
where $D, E \subseteq \mathbb{R}^2$ are open and

where $\psi : D \longrightarrow E$ is smooth and bijective.

Next: def'n of multivariable integration

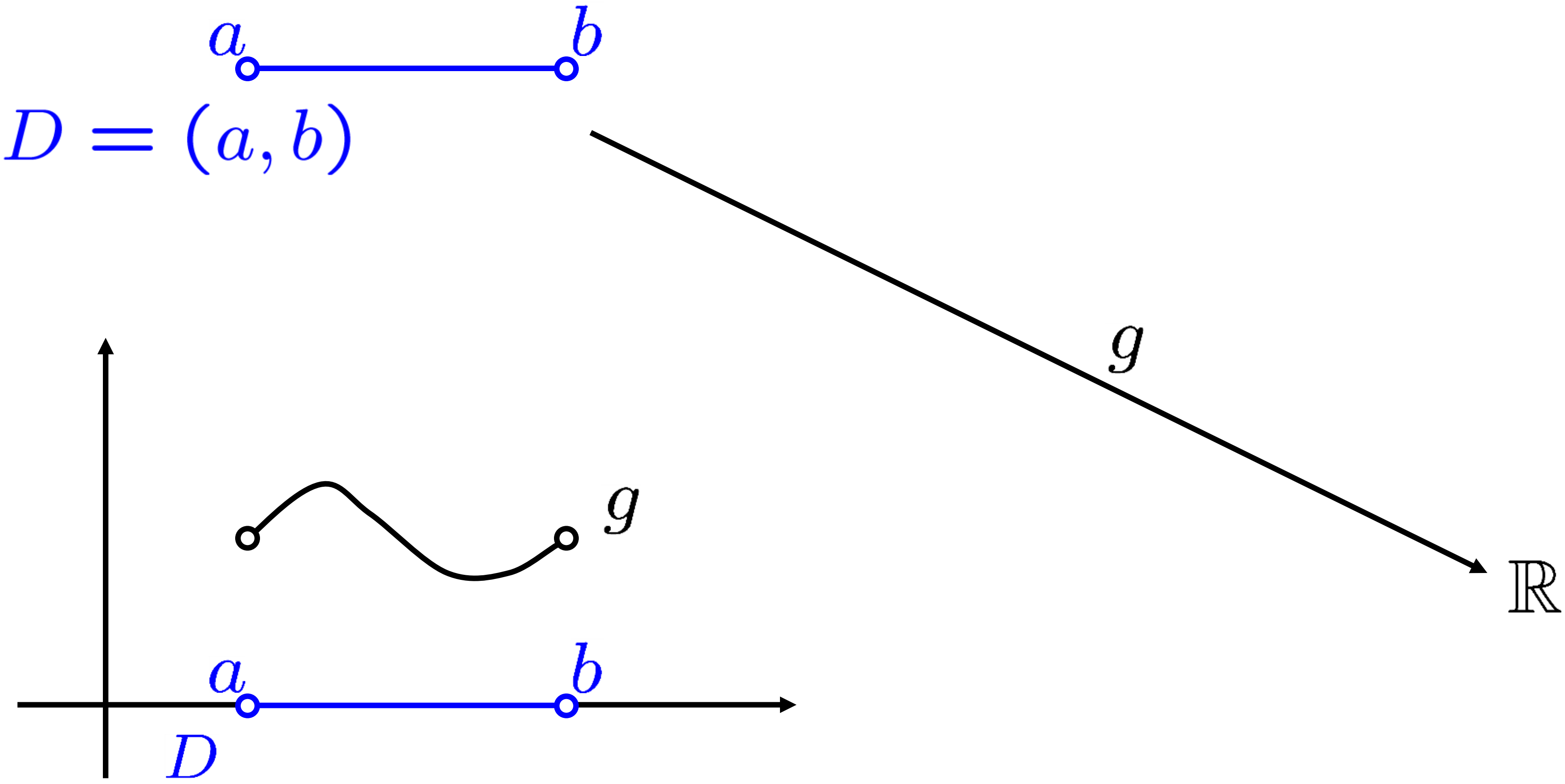
Definition of the Riemann integral (multivariable)

Recall the single-variable Riemann integral. . .

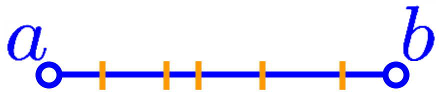


Graph of g lives "over" D.

$$\int\int_D g(s,t)\,ds\,dt = ?????$$



$$\int_D g(s)\, ds = ?????$$



$D = (a, b)$

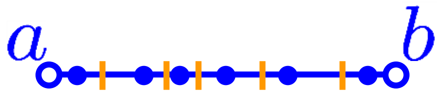
g

$\mathbb{R}$

Fix a small number $\delta > 0$.
Cover D by intervals of diameter $\leq \delta$.

(Not required, but often we use intervals of the same length.)

$$\int_D g(s)\, ds = ?????$$



$D = (a, b)$

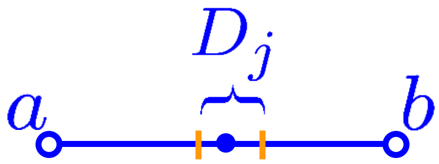
g

$\mathbb{R}$

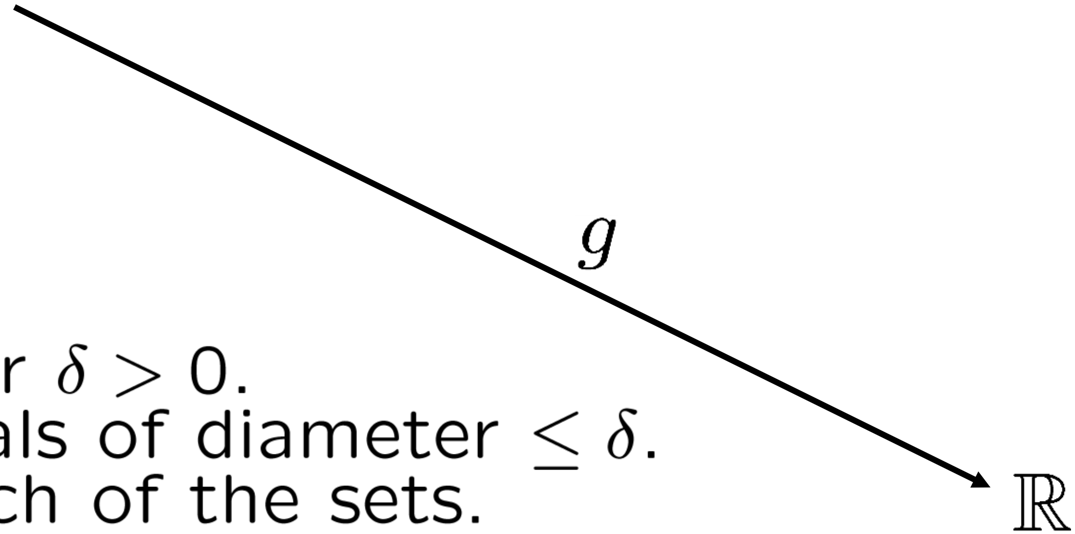
Fix a small number $\delta > 0$.
Cover D by intervals of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the jth, call it D_j.

(Not required, but often we take the midpoints.)

$$\int_D g(s)\, ds = ?????$$



$D = (a, b)$



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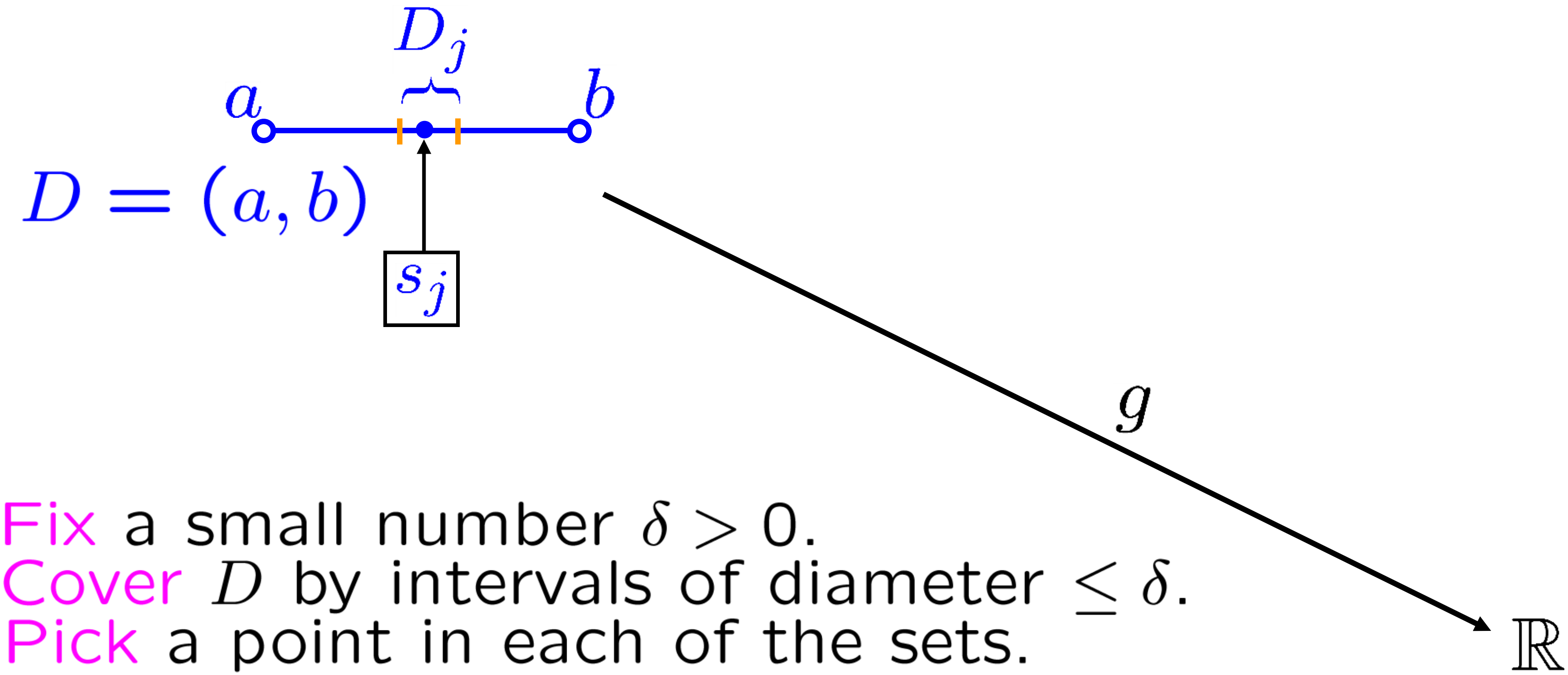

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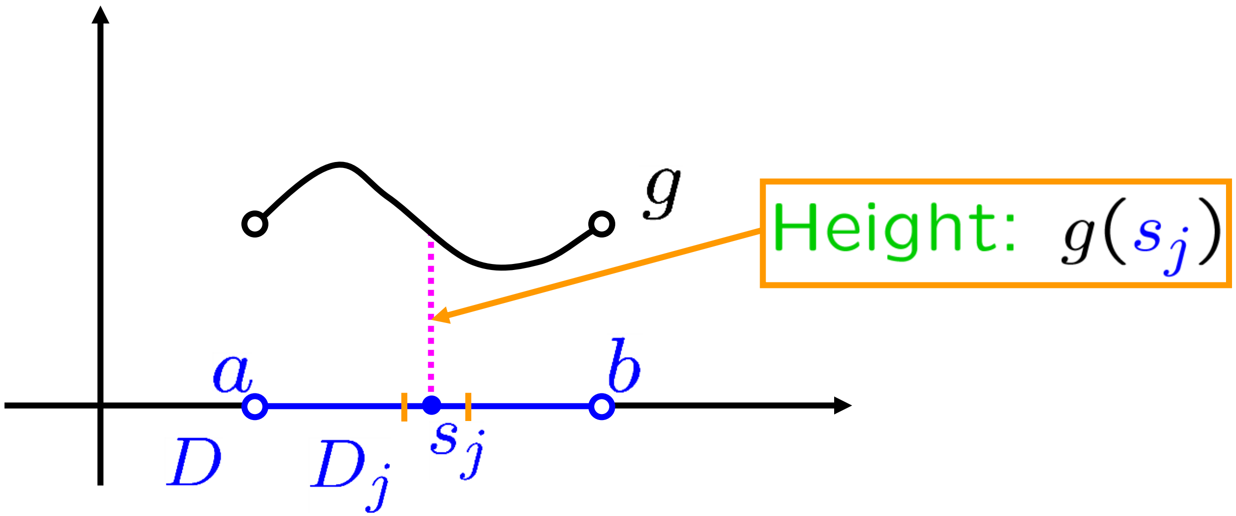

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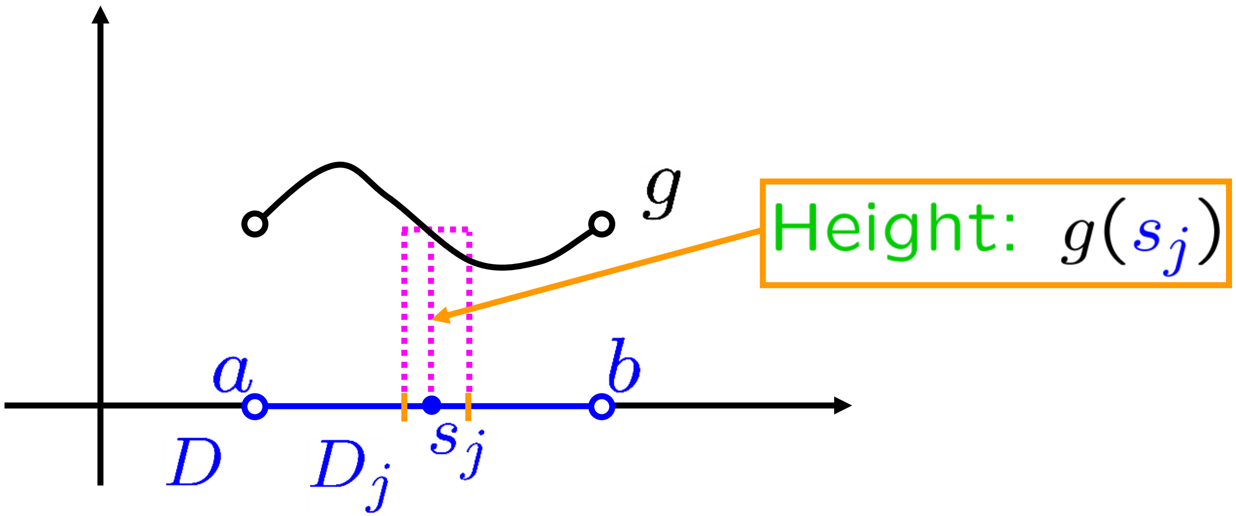

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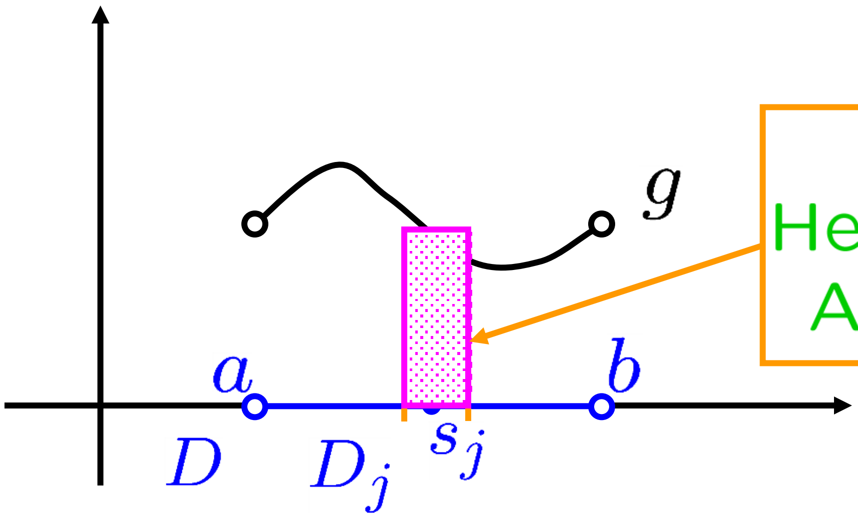
$$\int_D g(s)\,ds = ?????$$



Base: [Length(D_j)]
Height: $g(s_j)$
Area: $[g(s_j)]$[Length(D_j)]

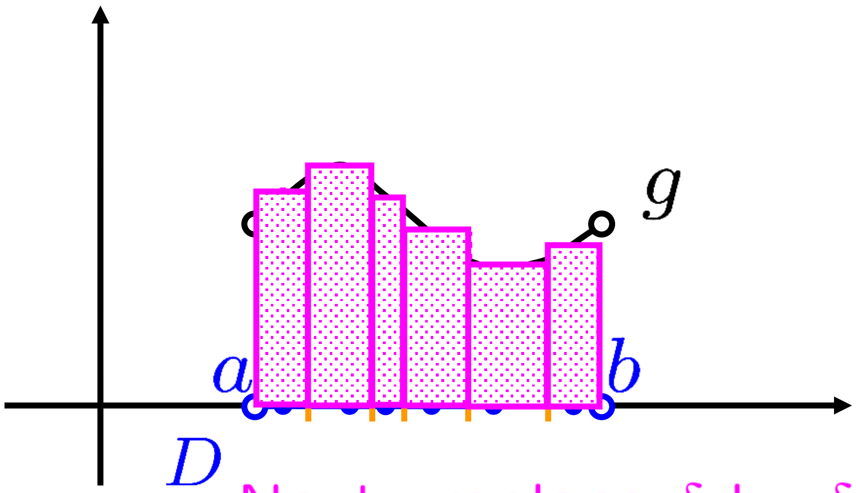
Fix a small number $\delta > 0$.
Cover D by intervals of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the jth, call it D_j.
Call its point s_j.

Compute $[g(s_j)]$[Length(D_j)].

$$\int_{D_j} g(s)\, ds \approx [g(s_j)][\text{Length}(D_j)]$$

Add over all j.

$$\int_{D} g(s)\, ds = ?????$$



Next: replace δ by $\delta_k \to 0$

Fix a small number $\delta > 0$.
Cover D by intervals of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the jth, call it D_j.
Call its point s_j.

Compute $[g(s_j)][\text{Length}(D_j)]$.

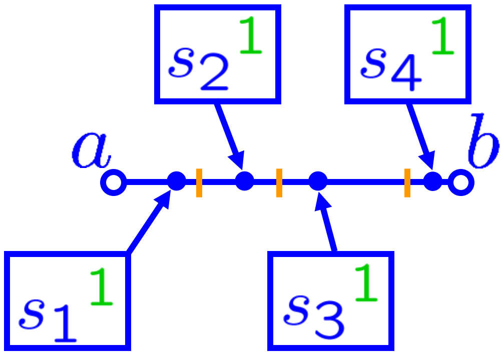
$$\int_{D_j} g(s)\,ds \approx [g(s_j)][\text{Length}(D_j)]$$

Add over all j.

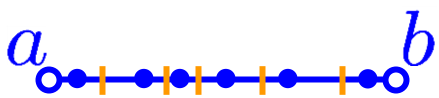
$$\int_D g(s)\,ds \approx \sum_j [g(s_j)][\text{Length}(D_j)]$$

Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

e.g.: $\delta_1 = 1/2$, $\delta_2 = 1/3$, $\delta_3 = 1/4, \ldots$

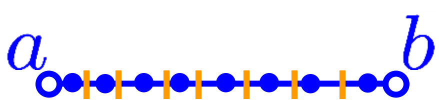


$D_1{}^1$, $D_2{}^1$, $D_3{}^1$, $D_4{}^1$
all of diam. $\leq \delta_1 = 1/2$



$s_1{}^2$, $s_2{}^2, \ldots s_6{}^2$

$D_1{}^2$, $D_2{}^2, \ldots D_6{}^2$
all of diam. $\leq \delta_2 = 1/3$



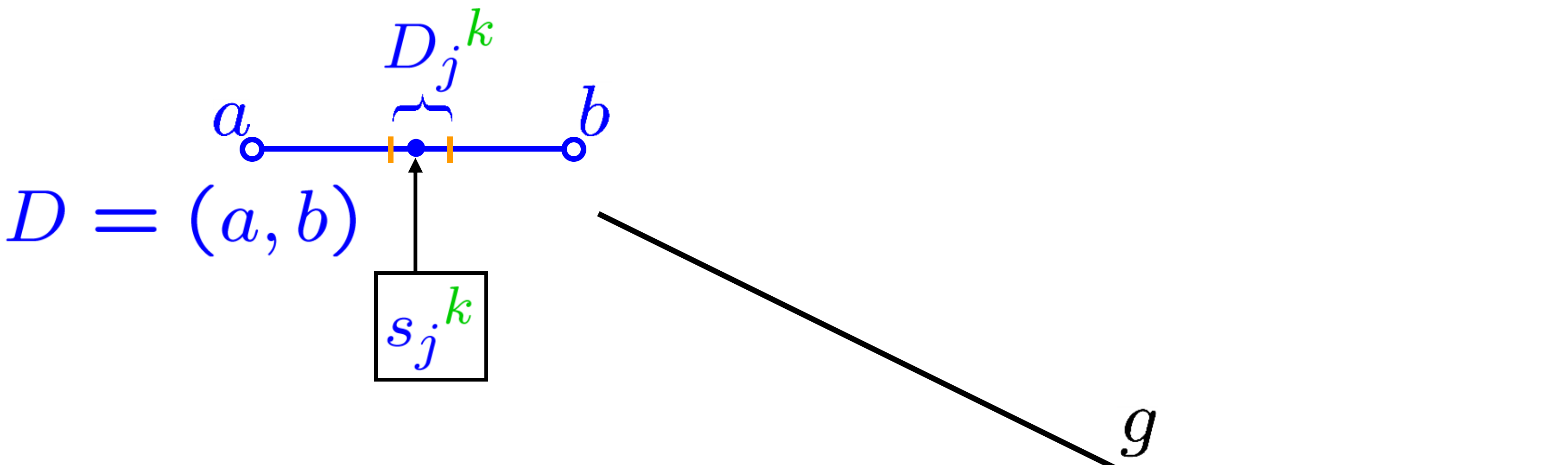
$D_1{}^3$, $D_2{}^3, \ldots D_8{}^3$
all of diam. $\leq \delta_3 = 1/4$

$\vdots$ $\vdots$

Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

$D = (a, b)$

Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.



Let $\delta_1, \delta_2, \ldots \to 0$ be pos. numbers.

$\forall k$, cover D by intervals of diameter $\leq \delta_k$.

$\forall k$, pick a point in each of the sets.

Focus on one k and one set, the jth, $D_j{}^k$.

Call its point $s_j{}^k$.

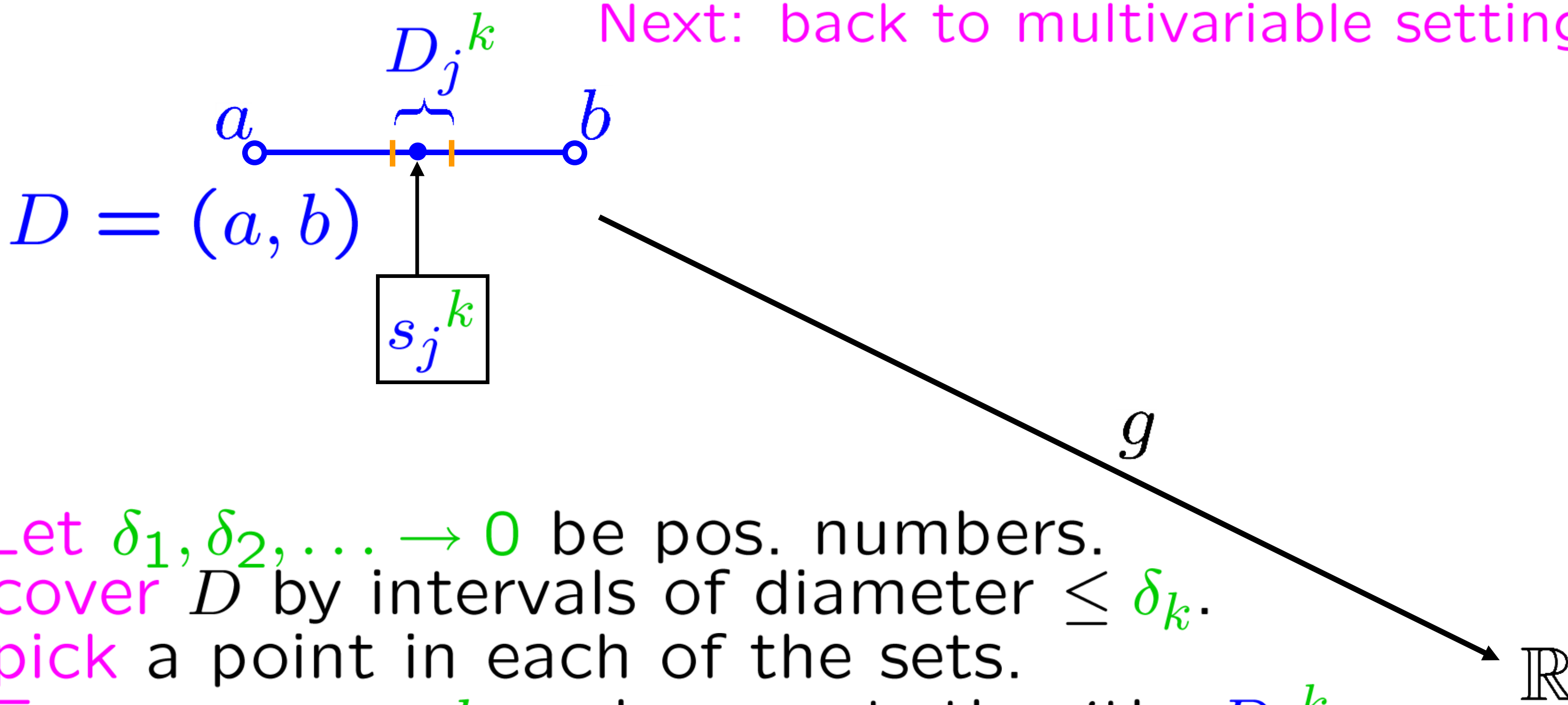
Compute $[g(s_j{}^k)][\text{Length}(D_j{}^k)]$.

$$\int_{D_j{}^k} g(s)\,ds \approx [g(s_j{}^k)][\text{Length}(D_j{}^k)].$$

Add over all j, and let $k \to \infty$.

$$\int_D g(s)\,ds \approx \sum_j [g(s_j{}^k)][\text{Length}(D_{j^k})]$$

Next: back to multivariable setting



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$\forall k$, pick a point in each of the sets.

Focus on one k and one set, the jth, D_j^k.

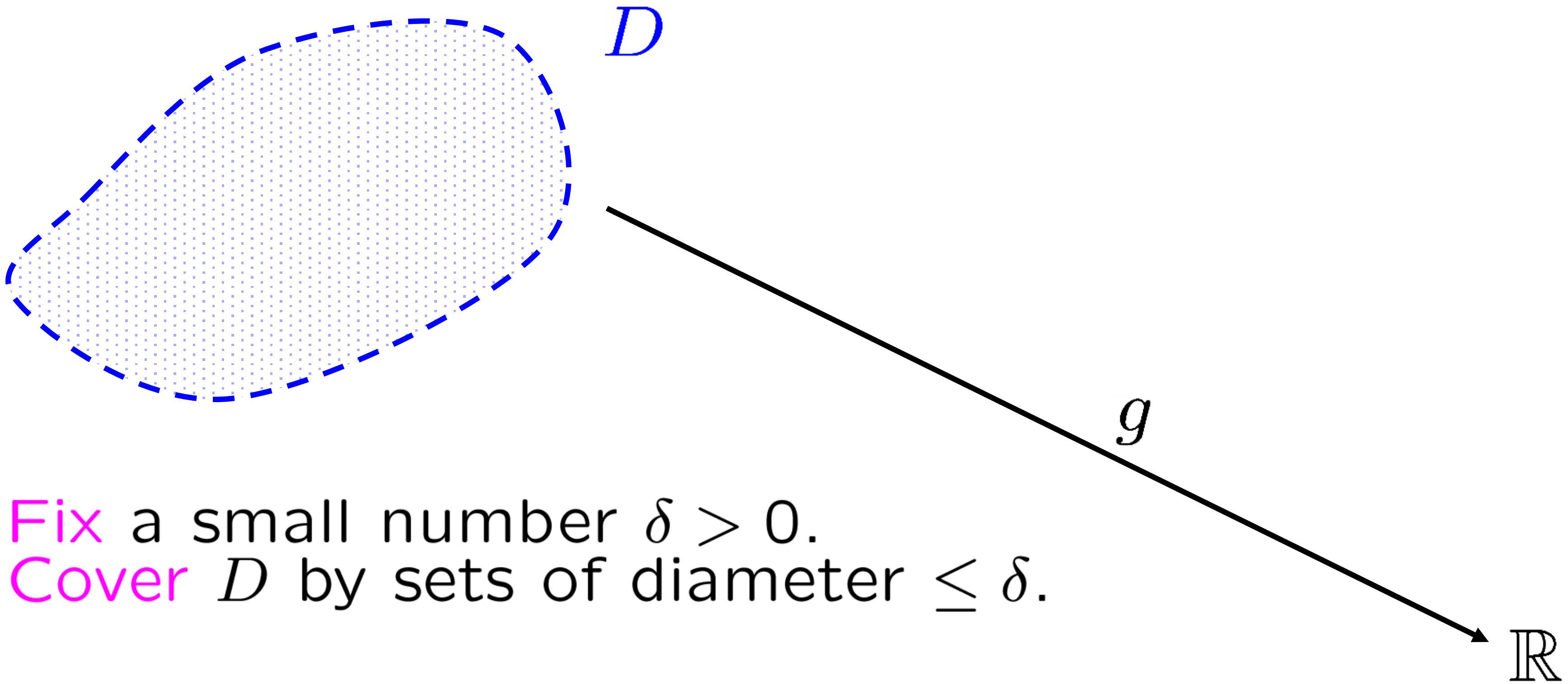
Call its point s_j^k.

Compute $[g(s_j^k)][\text{Length}(D_j^k)]$.

$$\int_{D_j^k} g(s)\, ds \approx [g(s_j^k)][\text{Length}(D_j^k)].$$

Add over all j, and let $k \to \infty$.

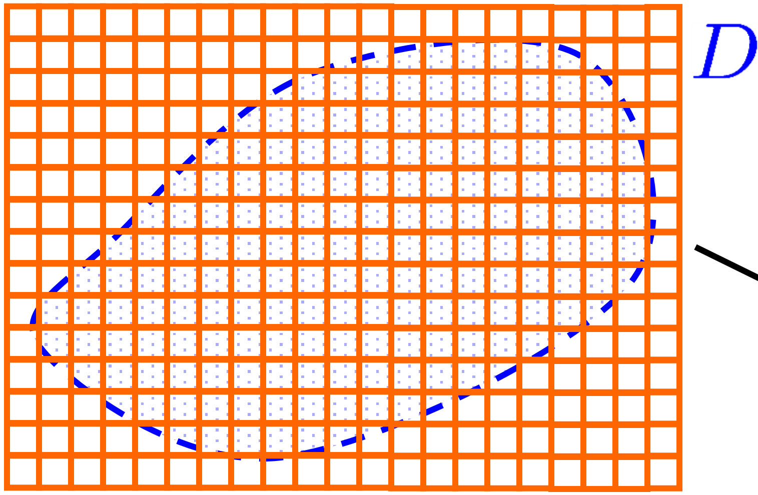
$$\boxed{\int_D g(s)\, ds} := \lim_{k\to\infty} \sum_j [g(s_j^k)][\text{Length}(D_{j^k})]$$

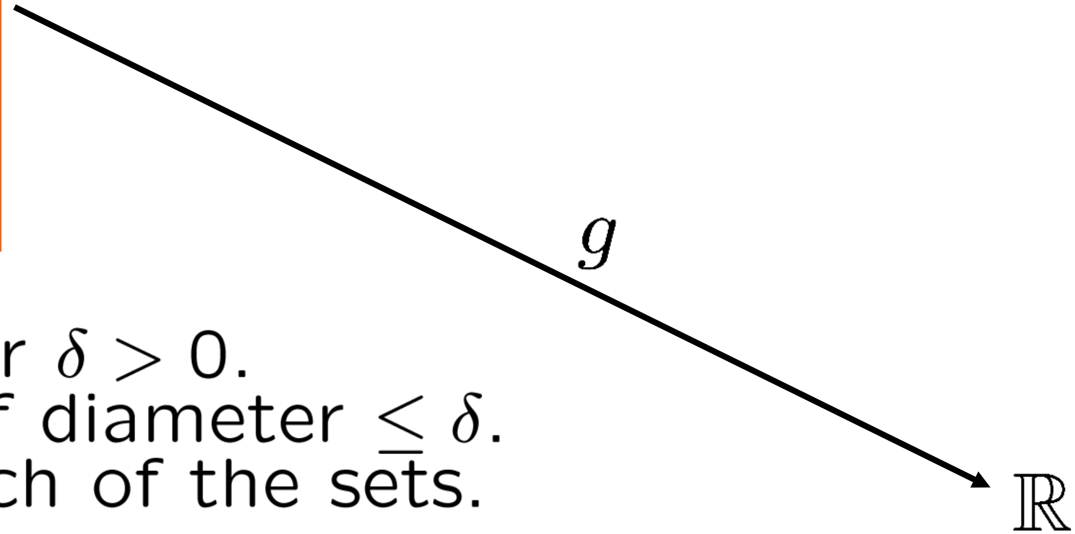


Fix a small number $\delta > 0$.
Cover D by sets of diameter $\leq \delta$.

(Not required, but helps if
the areas of the sets are
easily calculated, *e.g.*, squares.)

$$\int\int_D g(s,t)\, ds\, dt := ?????$$





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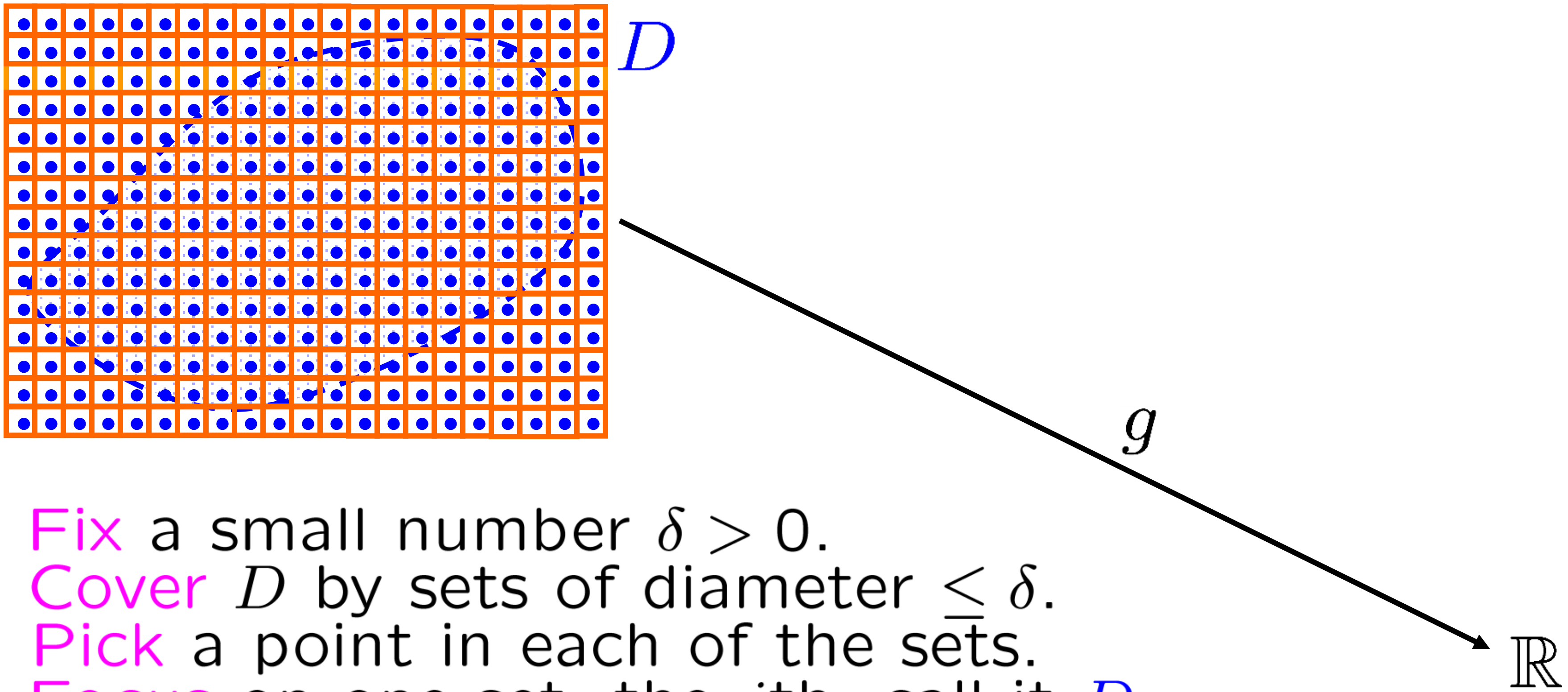
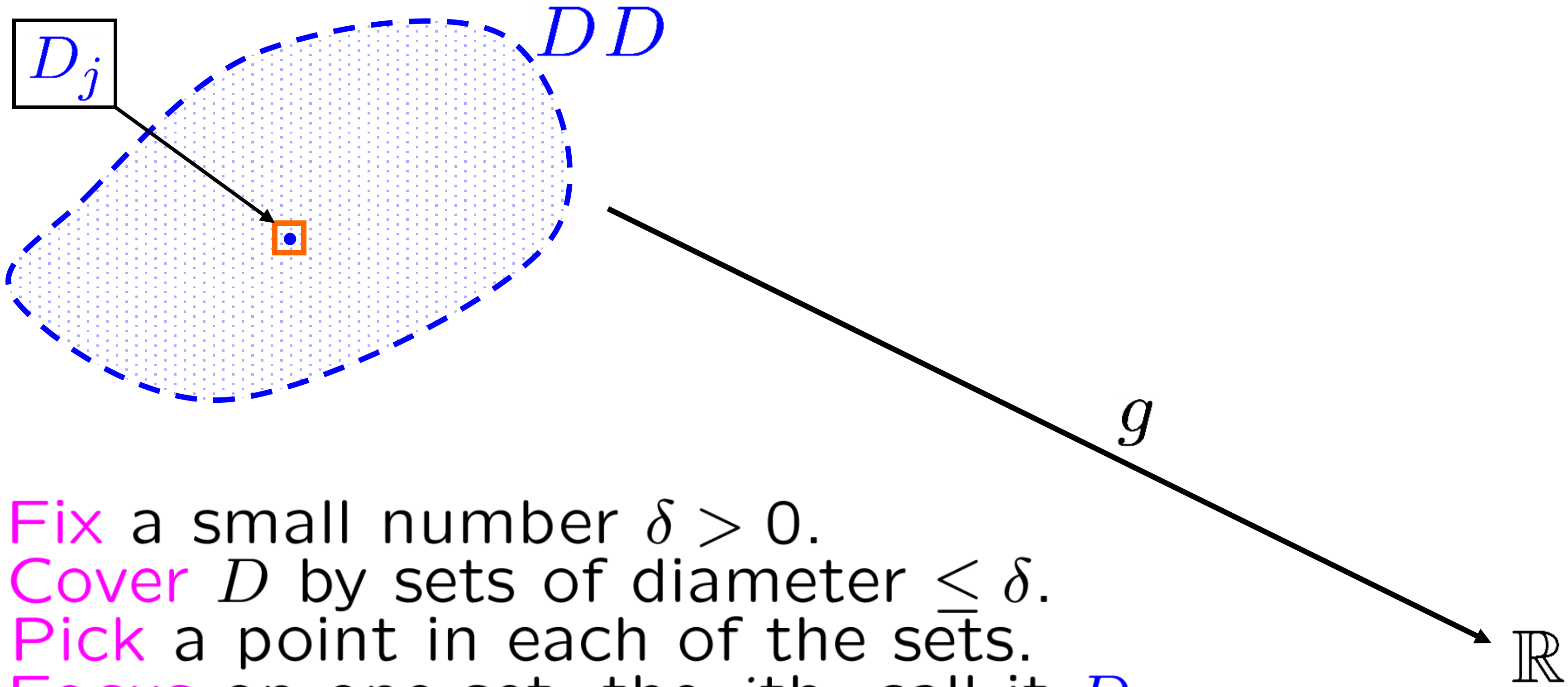


Fix a small number $\delta > 0$.
Cover D by sets of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the jth, call it D_j.

(Not required, but often
one takes the centers.)

$$\int \int_D g(s,t)\, ds\, dt := ?????$$



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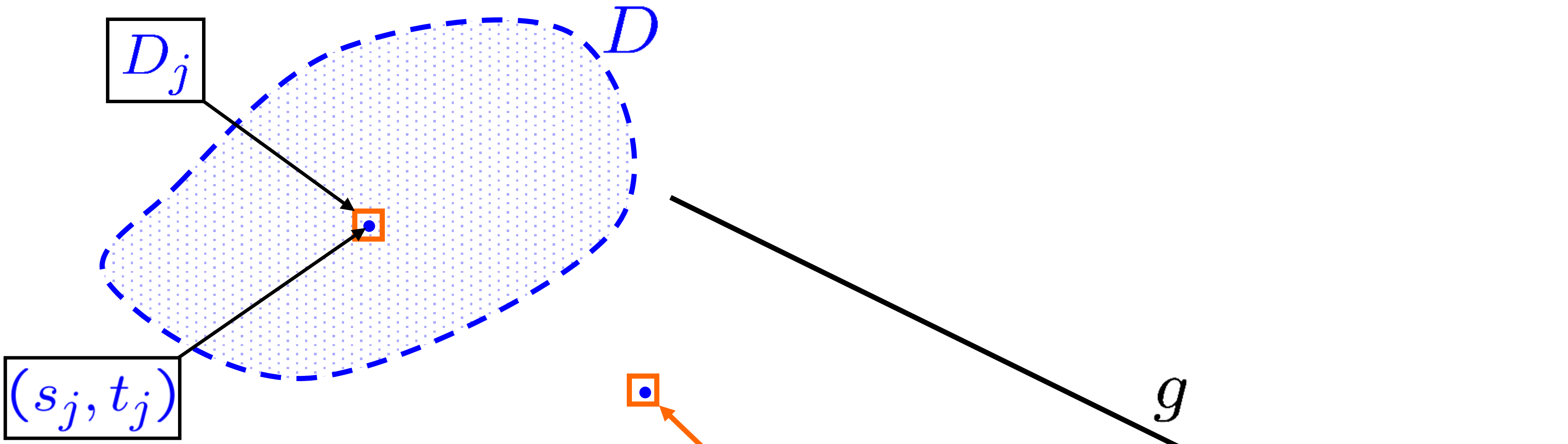


Fix a small number $\delta > 0$.
Cover D by sets of diameter $\leq \delta$.
Pick a point in each of the sets.
Focus on one set, the jth, call it D_j.
Call its point (s_j, t_j).

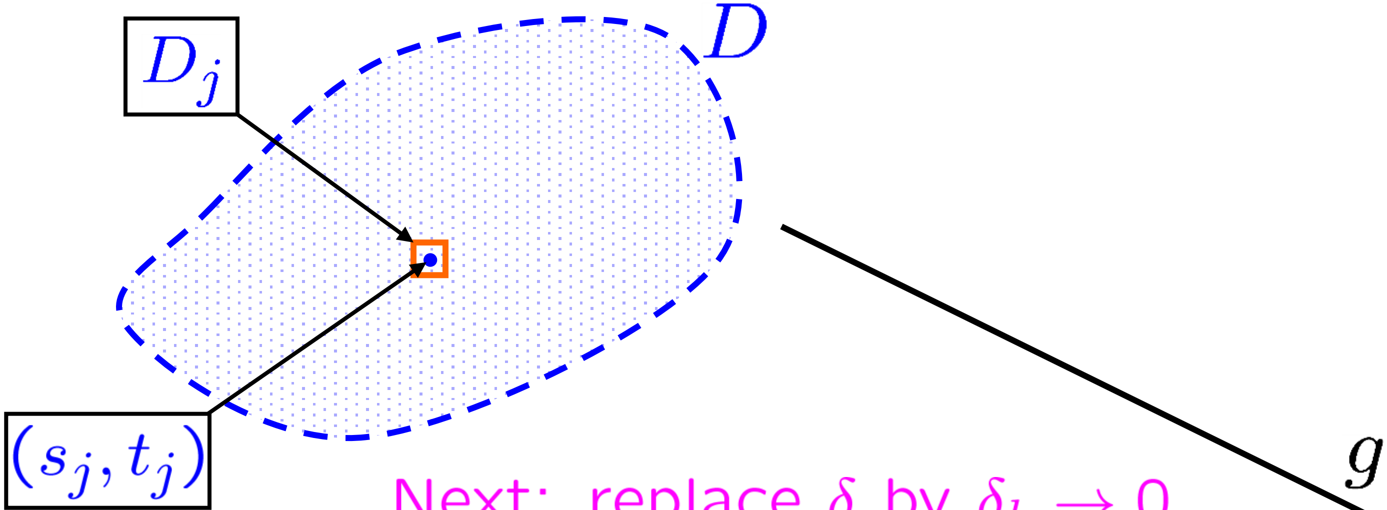
Compute $[g(s_j, t_j)][\text{Area}(D_j)]$.

$$\int\int_{D_j} g(s,t)\, ds\, dt \approx [g(s_j, t_j)][\text{Area}(D_j)]$$

0, whenever $(s_j, t_j) \notin D$

Add over all j.

$$\int\int_{D} g(s,t)\, ds\, dt := ?????$$



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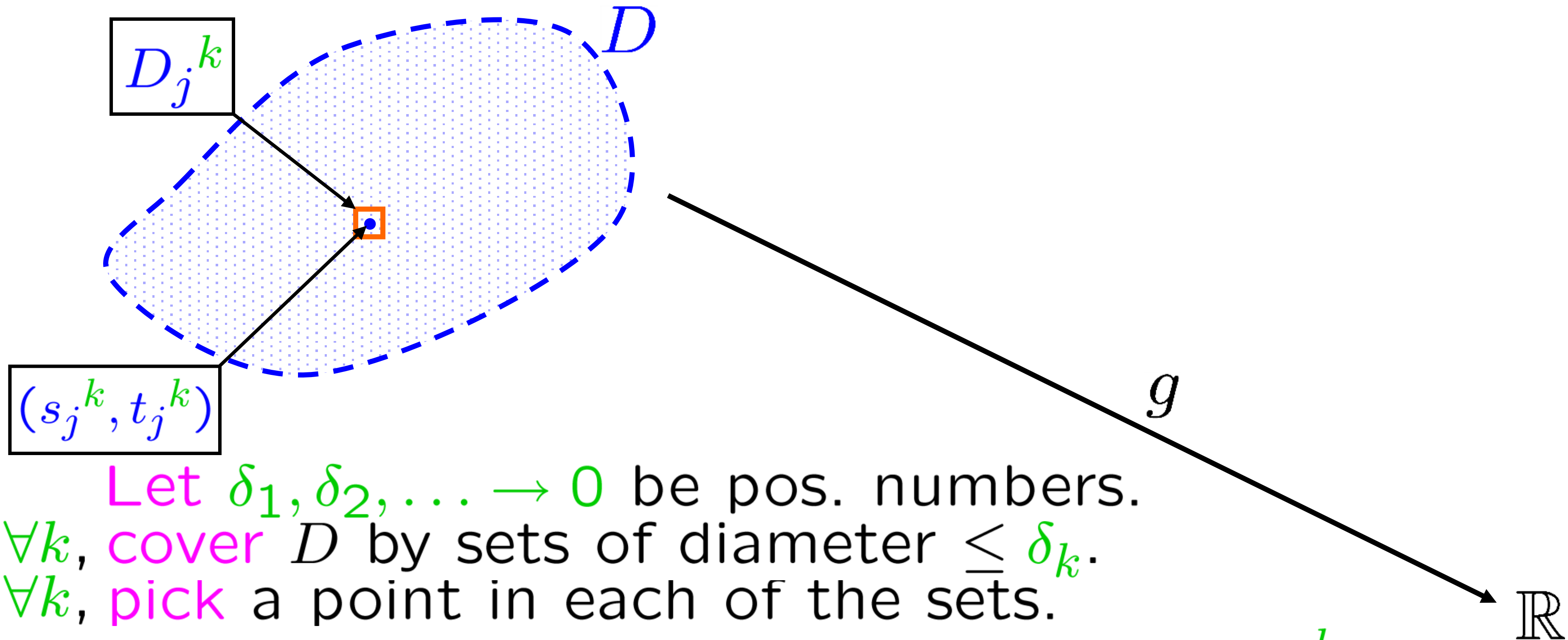
Add over all j.

$$\int\int_{D} g(s,t)\, ds\, dt \approx \sum_j [g(s_j, t_j)][\text{Area}(D_j)]$$



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$\forall k$, cover D by sets of diameter $\leq \delta_k$.

$\forall k$, pick a point in each of the sets.

Focus on one k and one set, the jth, $D_j{}^k$.

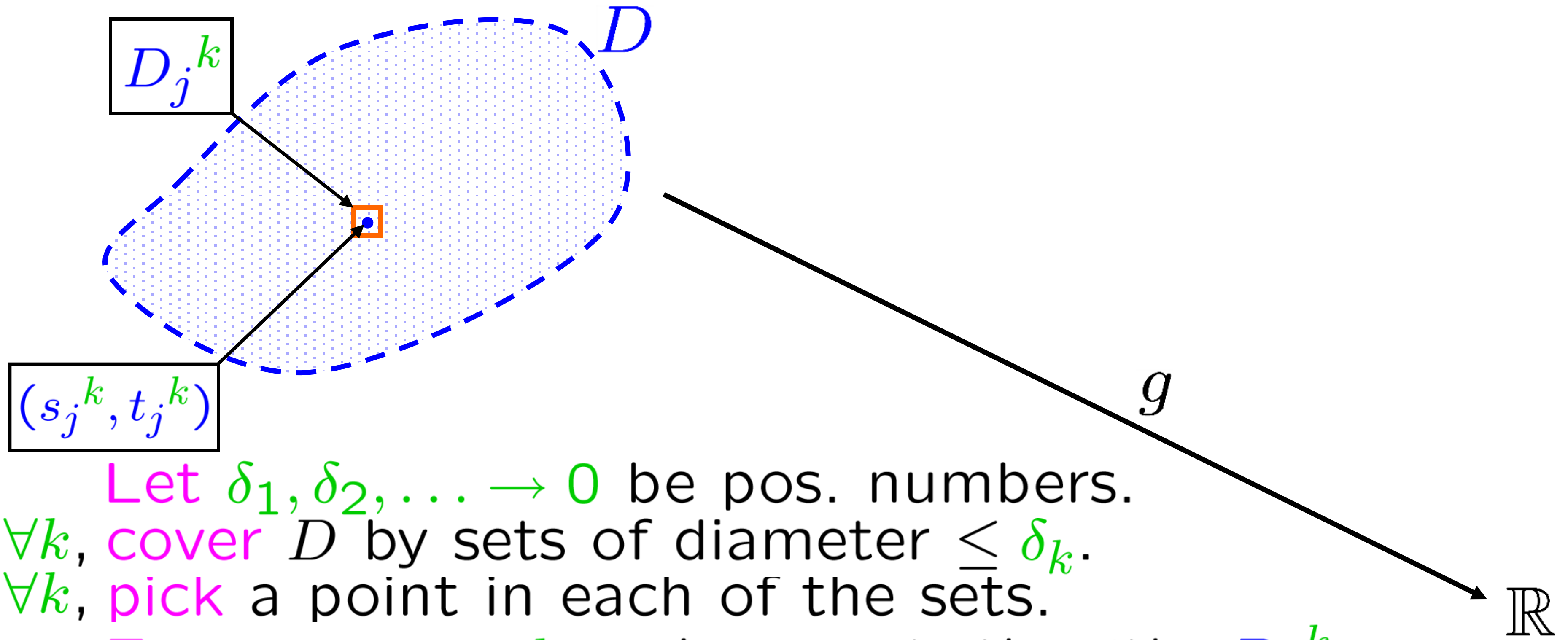
Call its point $(s_j{}^k, t_j{}^k)$.

Compute $[g(s_j{}^k, t_j{}^k)][\text{Area}(D_j{}^k)]$.

$$\int\int_{D_j{}^k} g(s,t)\, ds\, dt \approx [g(s_j{}^k, t_j{}^k)][\text{Area}(D_j{}^k)]$$

Add over all j, and let $k \to \infty$.

$$\int\int_{D} g(s,t)\, ds\, dt \approx \sum_j [g(s_j{}^k, t_j{}^k)][\text{Area}(D_j{}^k)]$$



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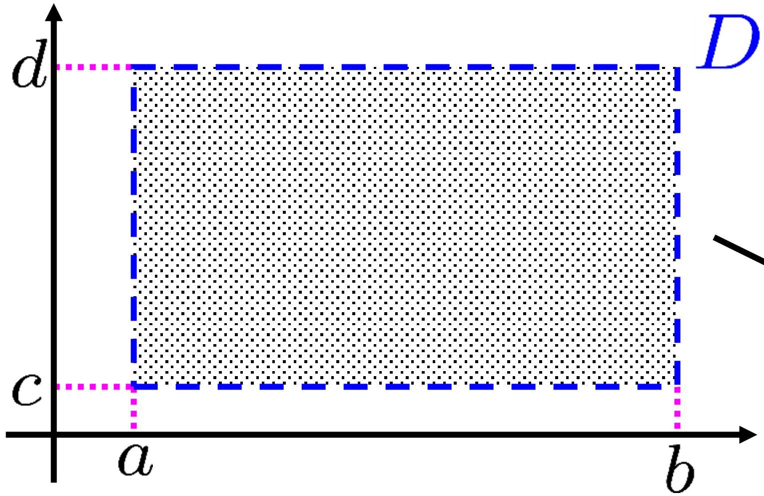
Add over all j, and let $k \to \infty$.

$$\int\int_D g(s,t)\, ds\, dt := \lim_{k\to\infty} \sum_j [g(s_j{}^k, t_j{}^k)][\text{Area}(D_j{}^k)]$$



extends to a continuous function on the closure $[a,b]\times[c,d]$

g

$\mathbb{R}$

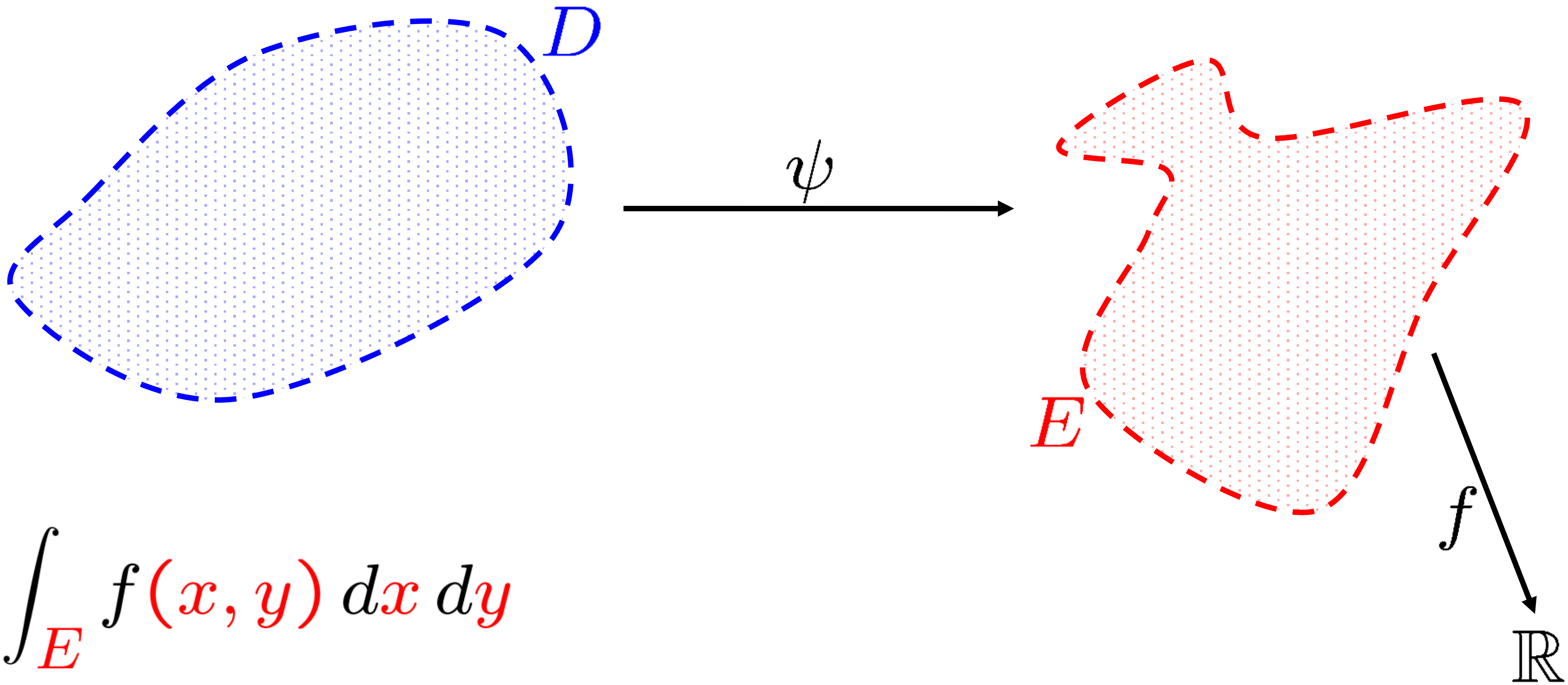
$$D = (a,b)\times(c,d)$$

Fubini's Theorem on rectangles:

$$\int\!\!\int_D g(s,t)\,ds\,dt \quad \text{exists!}$$

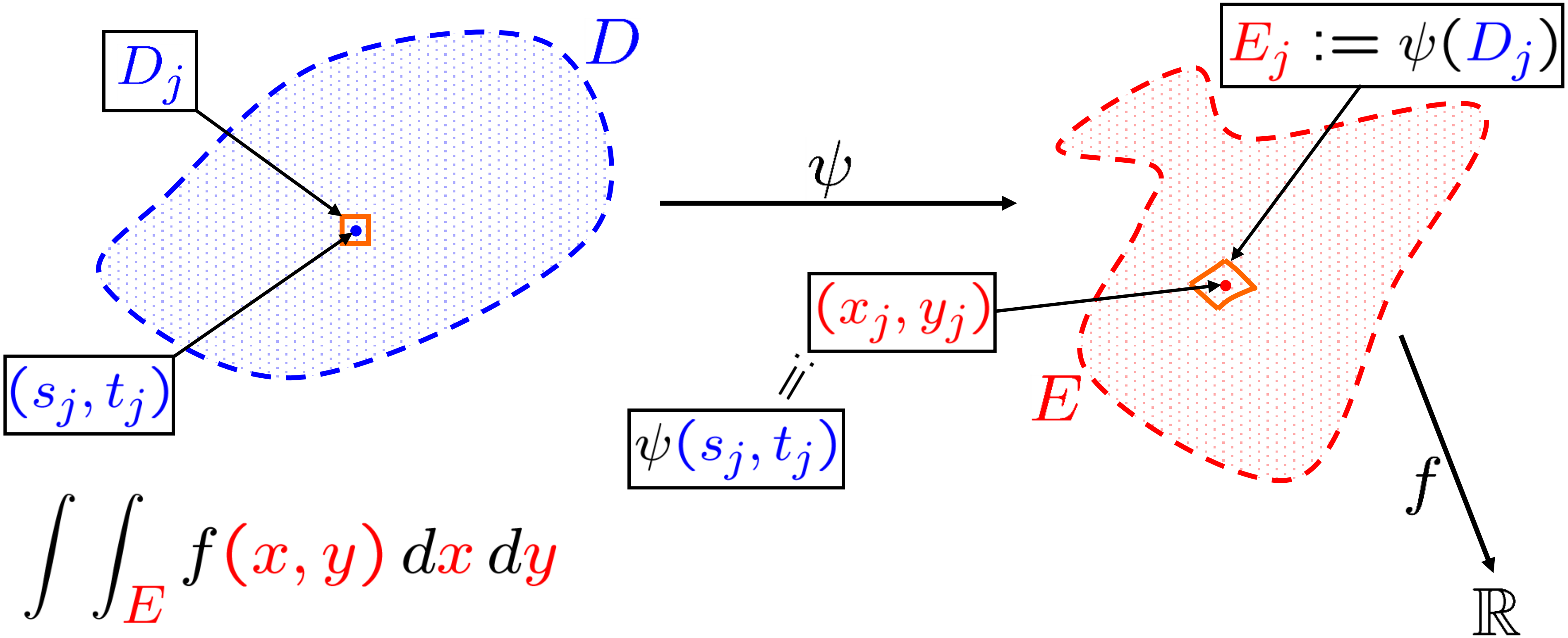
$$\int_a^b \int_c^d g(s,t)\,dt\,ds = \int\!\!\int_D g(s,t)\,ds\,dt = \int_c^d \int_a^b g(s,t)\,ds\,dt$$

Proof: We omit existence proof. For the equalities, use rectangular partitions and "follow your nose". QED



$$\int\int_E f(x,y)\,dx\,dy$$

$$= \int\int_D [f(\psi(s,t))][?????]\,ds\,dt$$

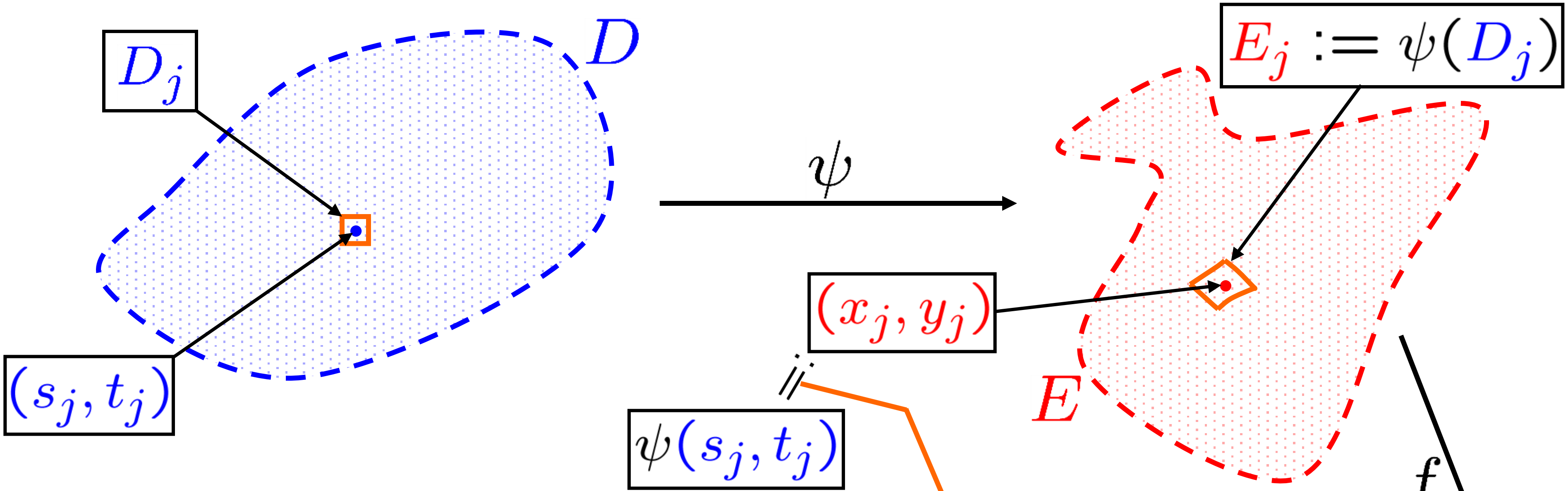


$$\int\int_E f(x,y)\,dx\,dy$$

$$\approx \sum_j [f(x_j, y_j)][\text{Area}(E_j)]$$

WHY?

$$\overset{?}{=} \int\int_D [f(\psi(s,t))][\,|\det(\psi'(s,t))|\,]\,ds\,dt$$



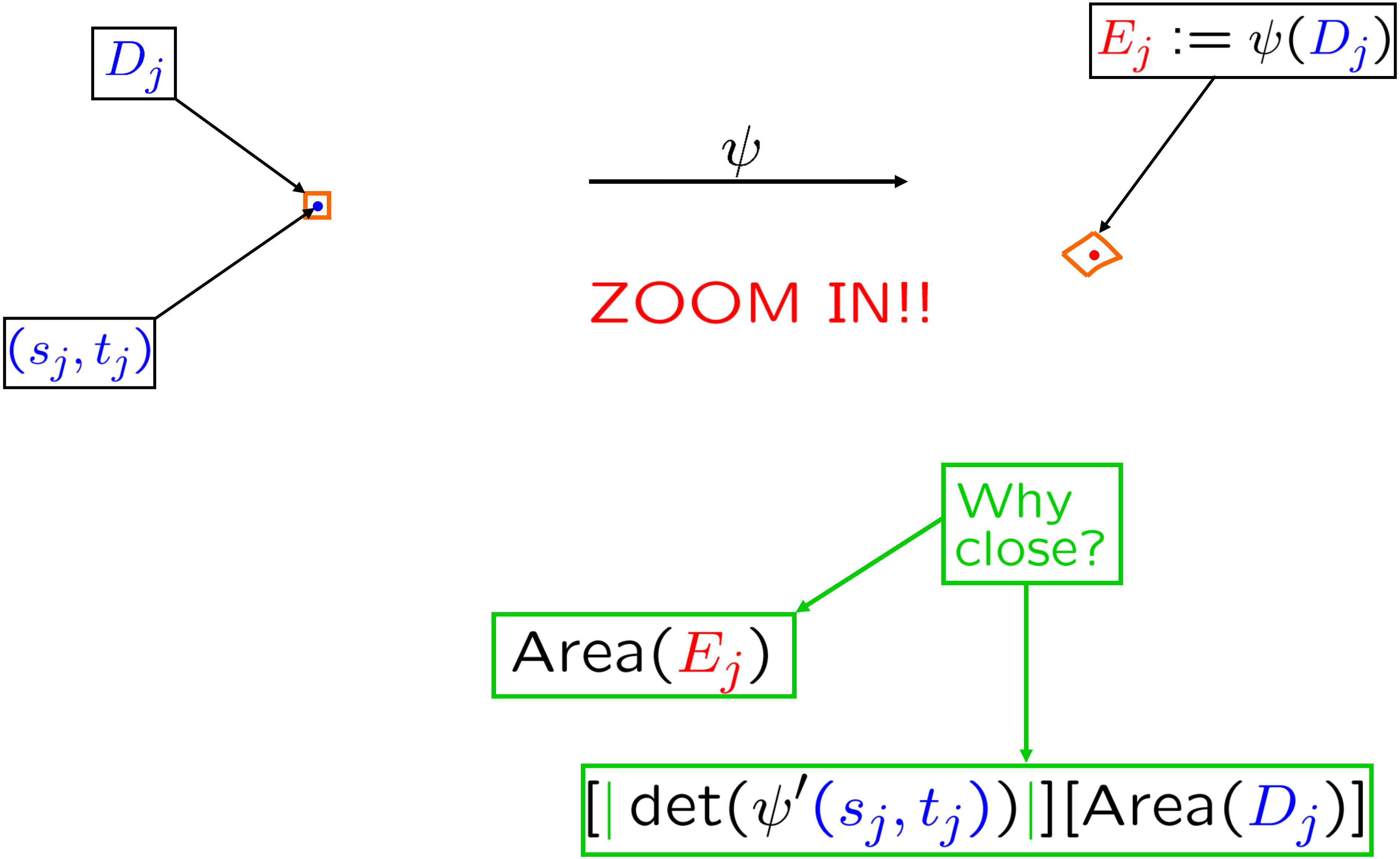
$$\int\int_E f(x,y)\,dx\,dy$$

$$\approx \sum_j [f(x_j,y_j)][\text{Area}(E_j)]$$

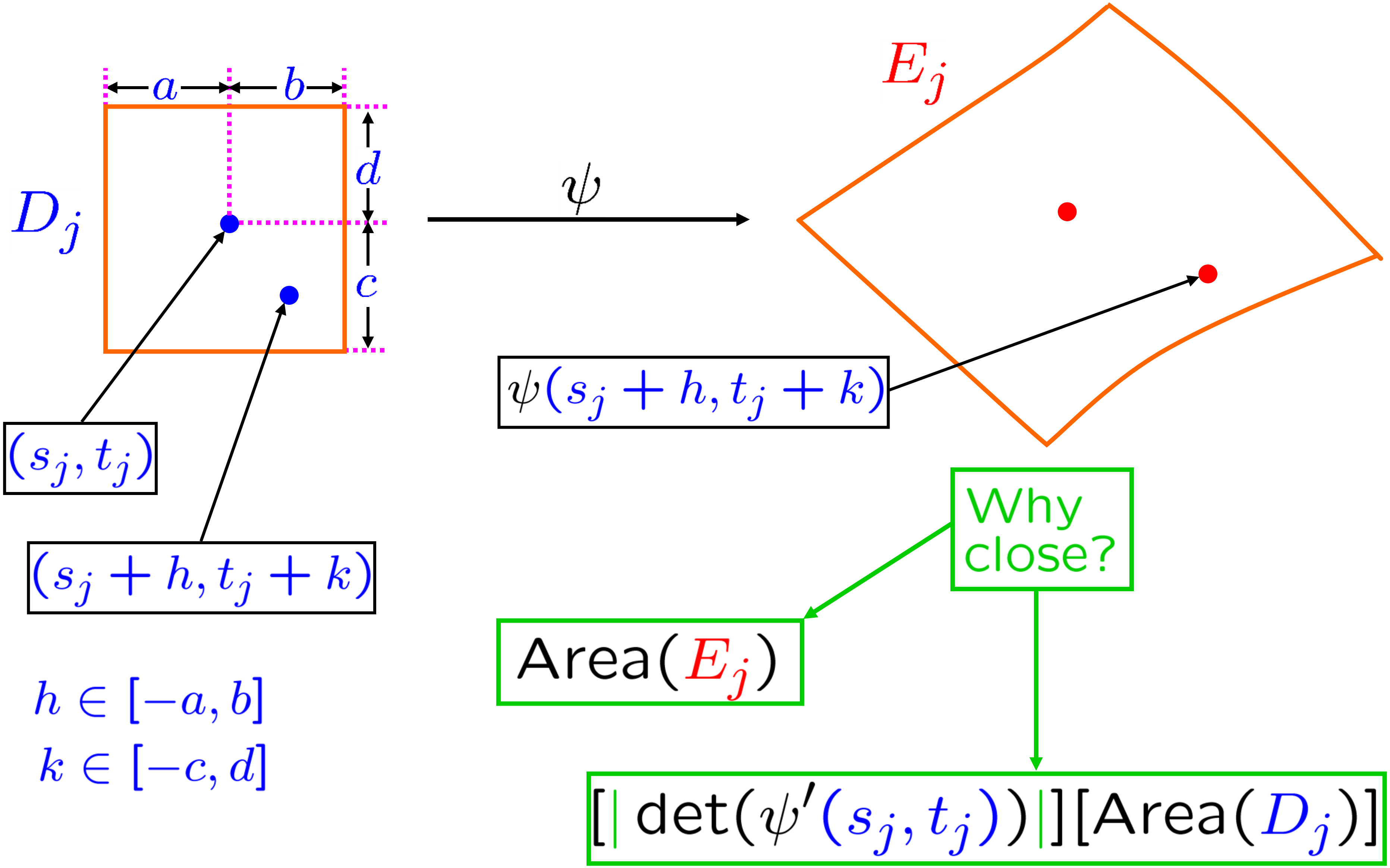
Why close?

$$\overset{?}{\approx} \sum_j [f(\psi(s_j,t_j))][|\det(\psi'(s_j,t_j))|][\text{Area}(D_j)]$$

$$\overset{?}{\approx} \int\int_D [f(\psi(s,t))][|\det(\psi'(s,t))|]\,ds\,dt$$

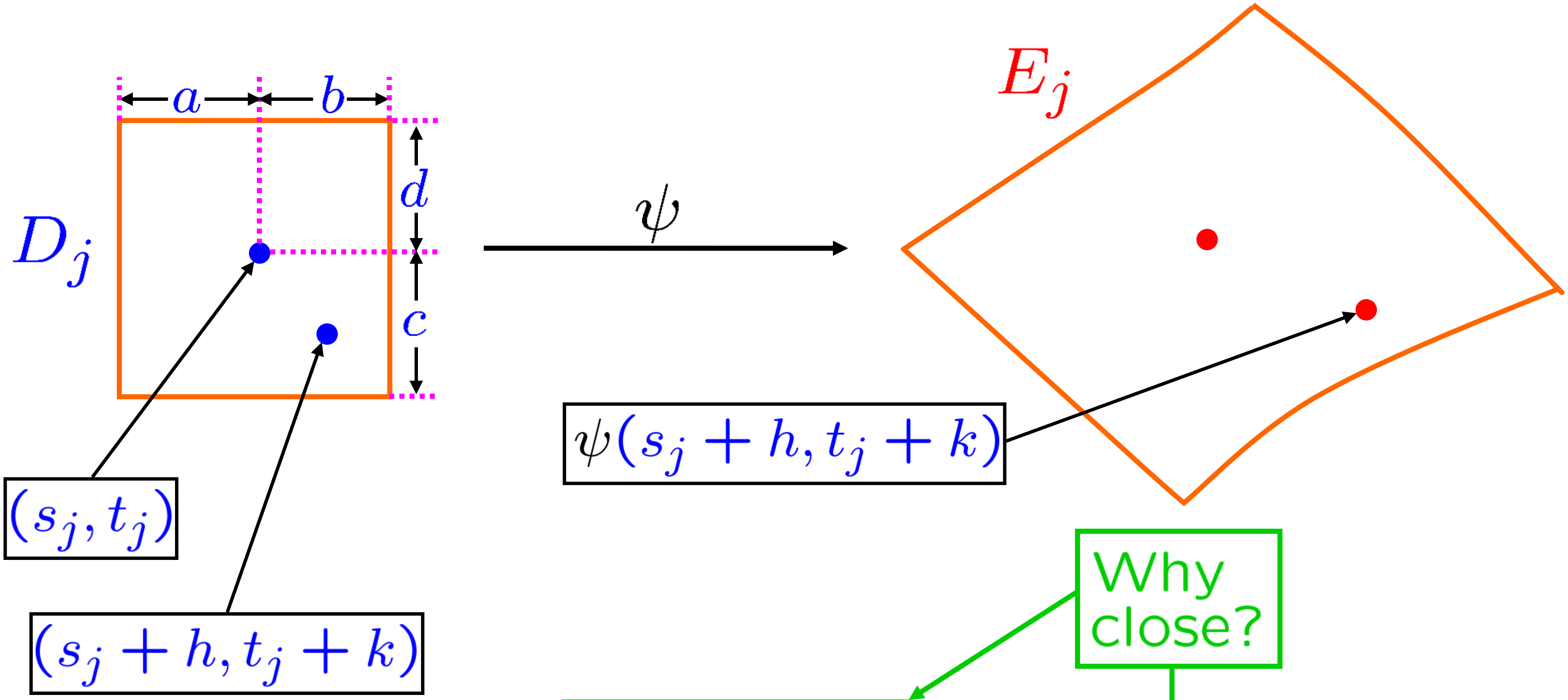


D_j
(s_j, t_j)
ψ
$E_j := \psi(D_j)$
ZOOM IN!!
Why close?
Area(E_j)
[| det($\psi'(s_j, t_j)$)|][Area(D_j)]



$$\psi(s_j + h, t_j + k) \approx [\psi(s_j, t_j)] + L_{\psi'(s_j,t_j)}(h, k)$$

$$E_j \approx [\psi(s_j, t_j)] + L_{\psi'(s_j,t_j)}([-a, b] \times [-c, d])$$



$h \in [-a, b]$

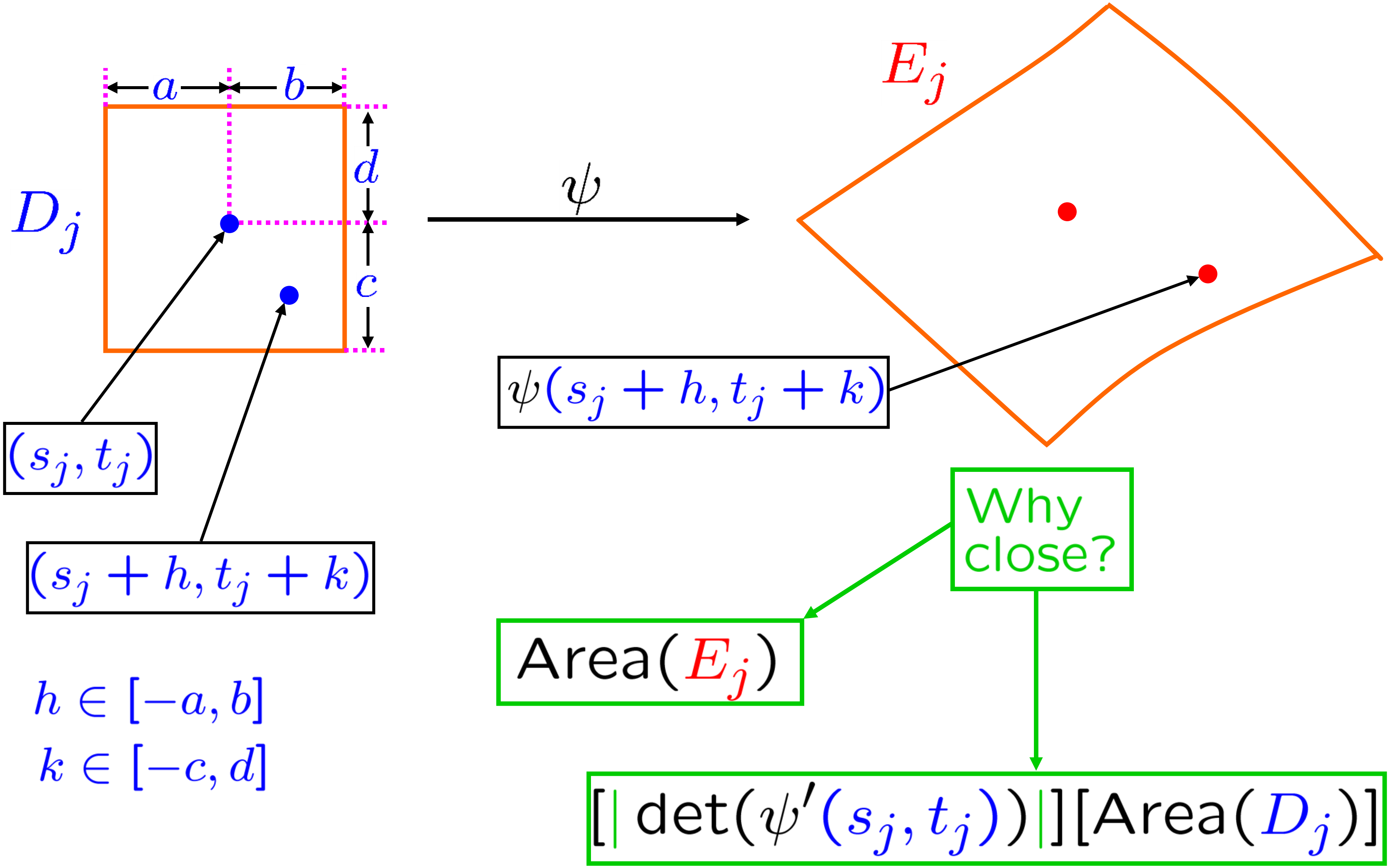
$k \in [-c, d]$

$\text{Area}(E_j)$

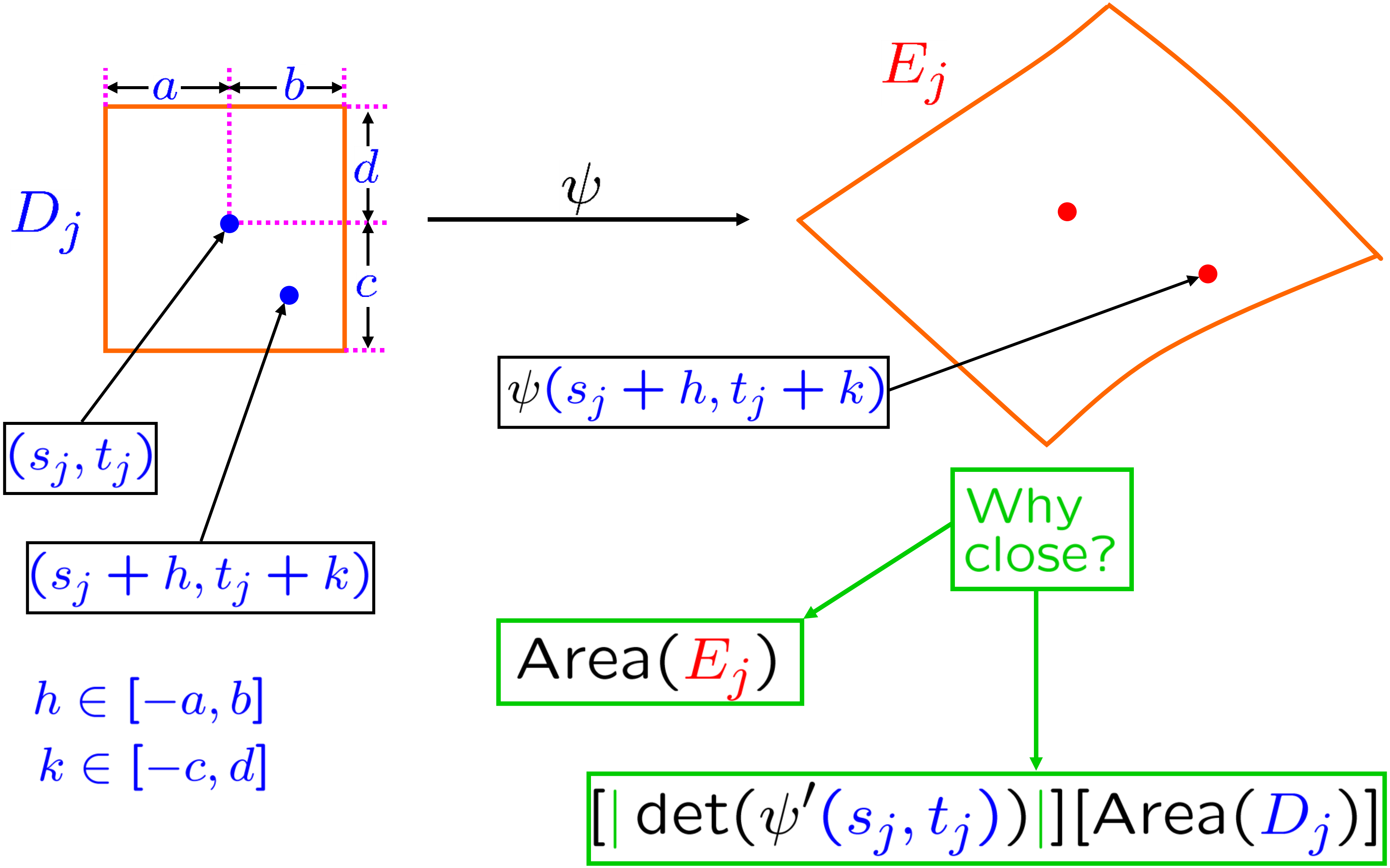
$[|\det(\psi'(s_j, t_j))|][\text{Area}(D_j)]$

$$\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d]))$$

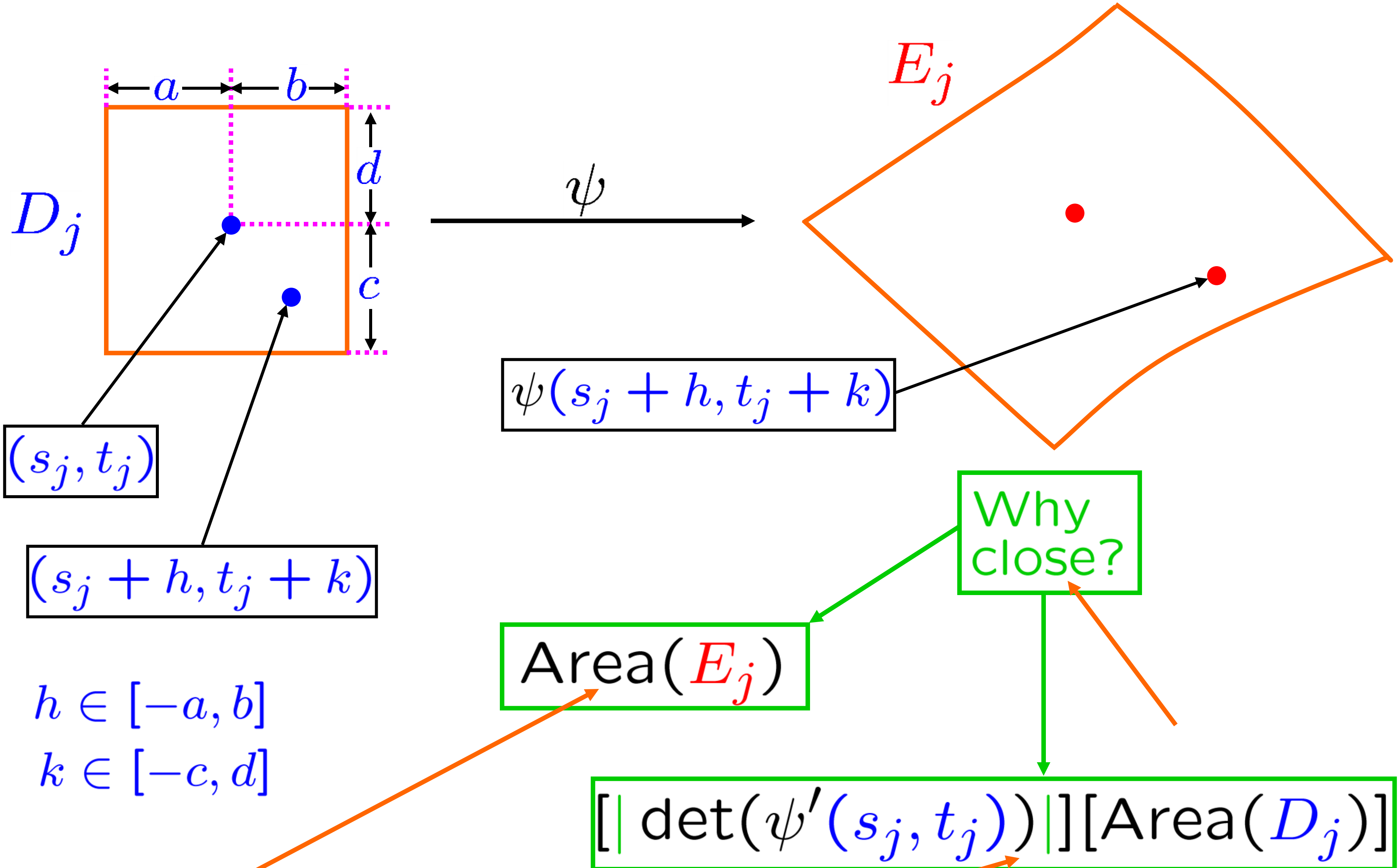
$$E_j \approx [\psi(s_j, t_j)] + L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d])$$



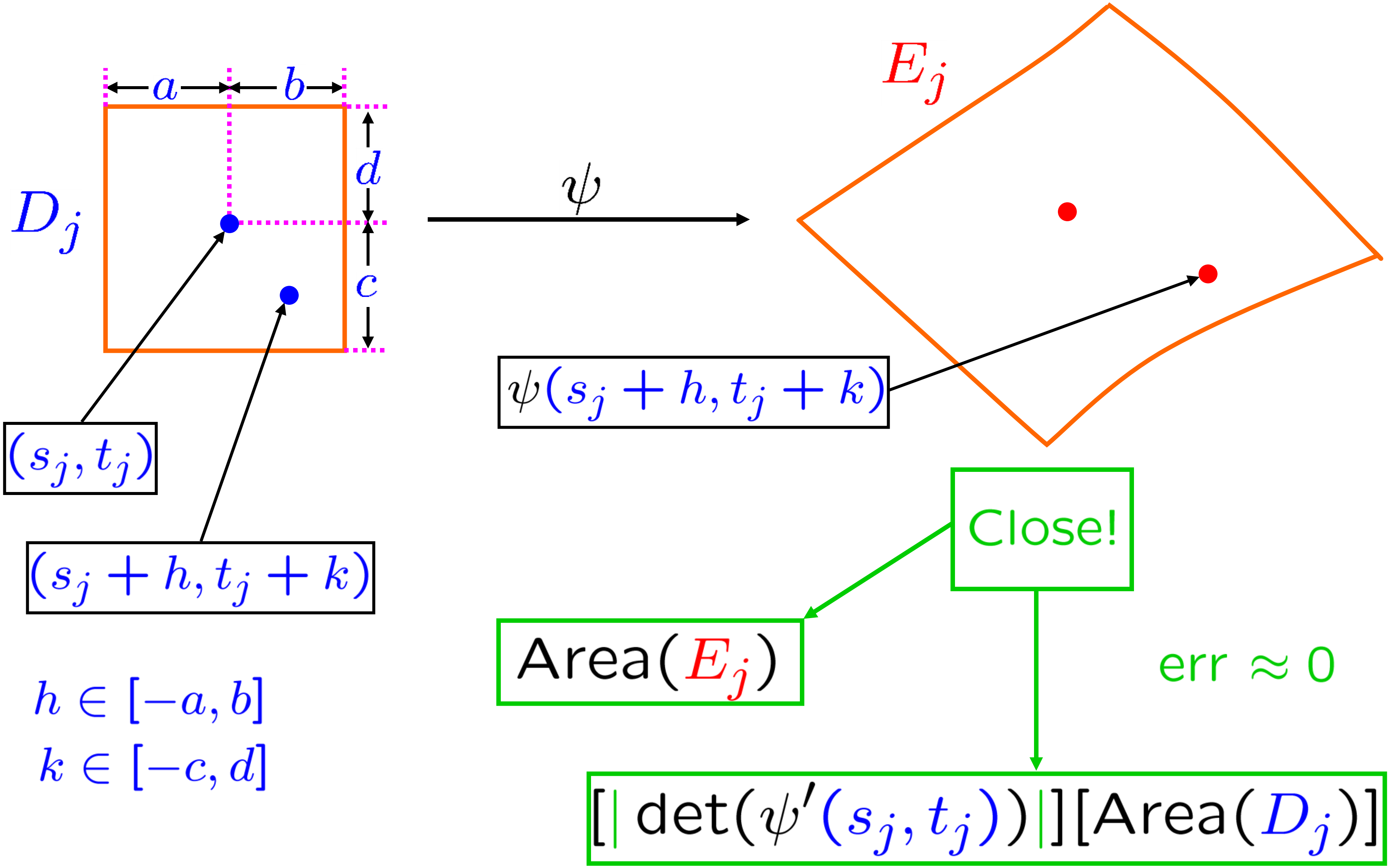
$$\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d]))$$
$$= [|\det(\psi'(s_j,t_j))|][\text{Area}([-a,b] \times [-c,d])]$$



$$\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d]))$$
$$= [|\det(\psi'(s_j, t_j))|][(a+b)(c+d)]$$



$$\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d]))$$
$$= [|\det(\psi'(s_j, t_j))|][\text{Area}(D_j)]$$



$$\text{Area}(E_j) \approx \text{Area}(L_{\psi'(s_j,t_j)}([-a,b] \times [-c,d]))$$
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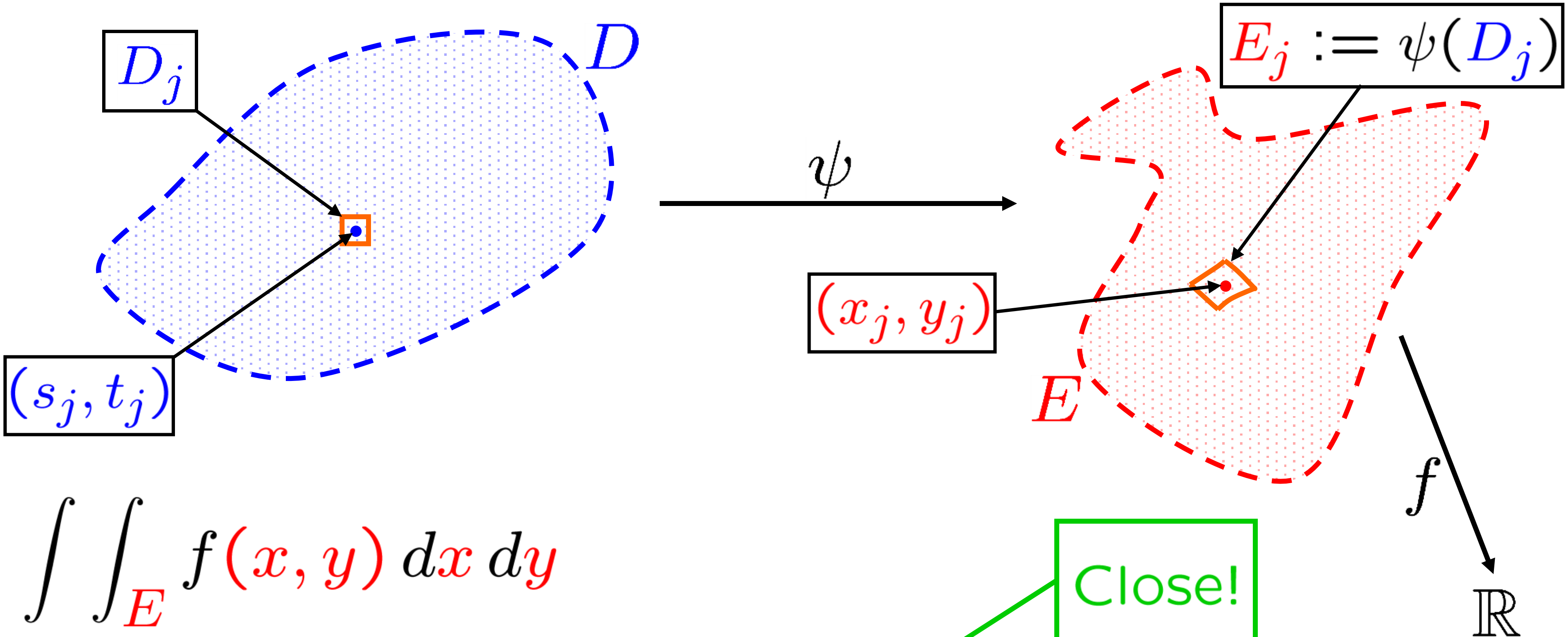


$$\int\int_E f(x, y)\, dx\, dy$$

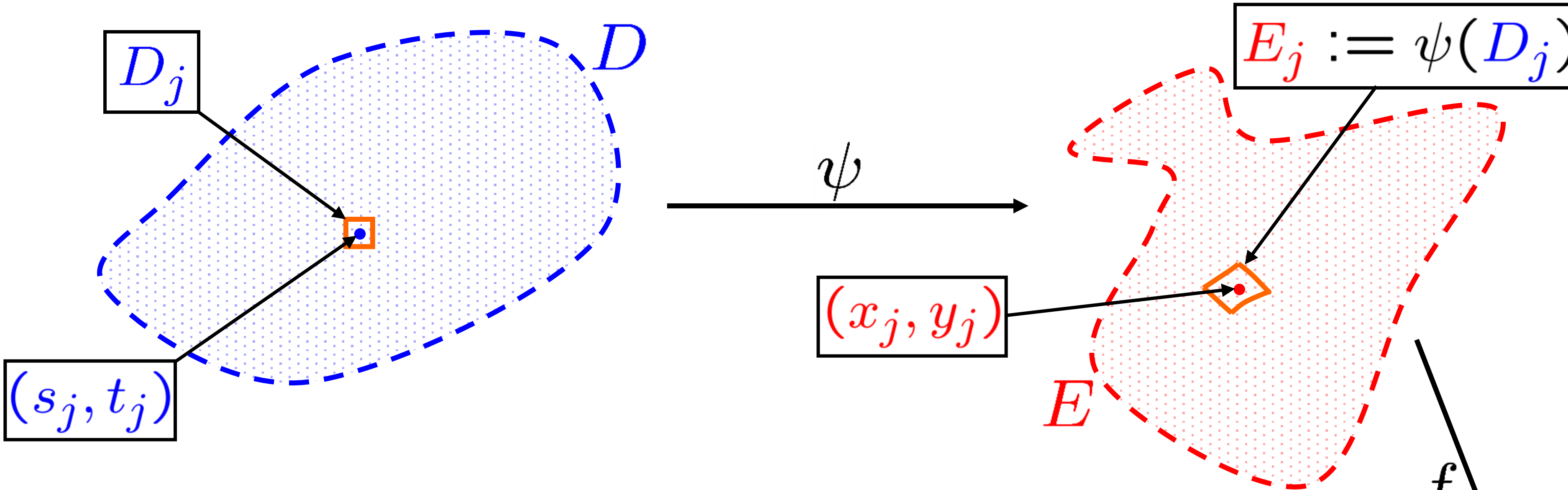
Close!

$$\approx \sum_j [f(x_j, y_j)][\text{Area}(E_j)]$$

$\sum \text{err} \approx 0$

$$\approx \sum_j [f(\psi(s_j, t_j))][|\det(\psi'(s_j, t_j))|][\text{Area}(D_j)]$$

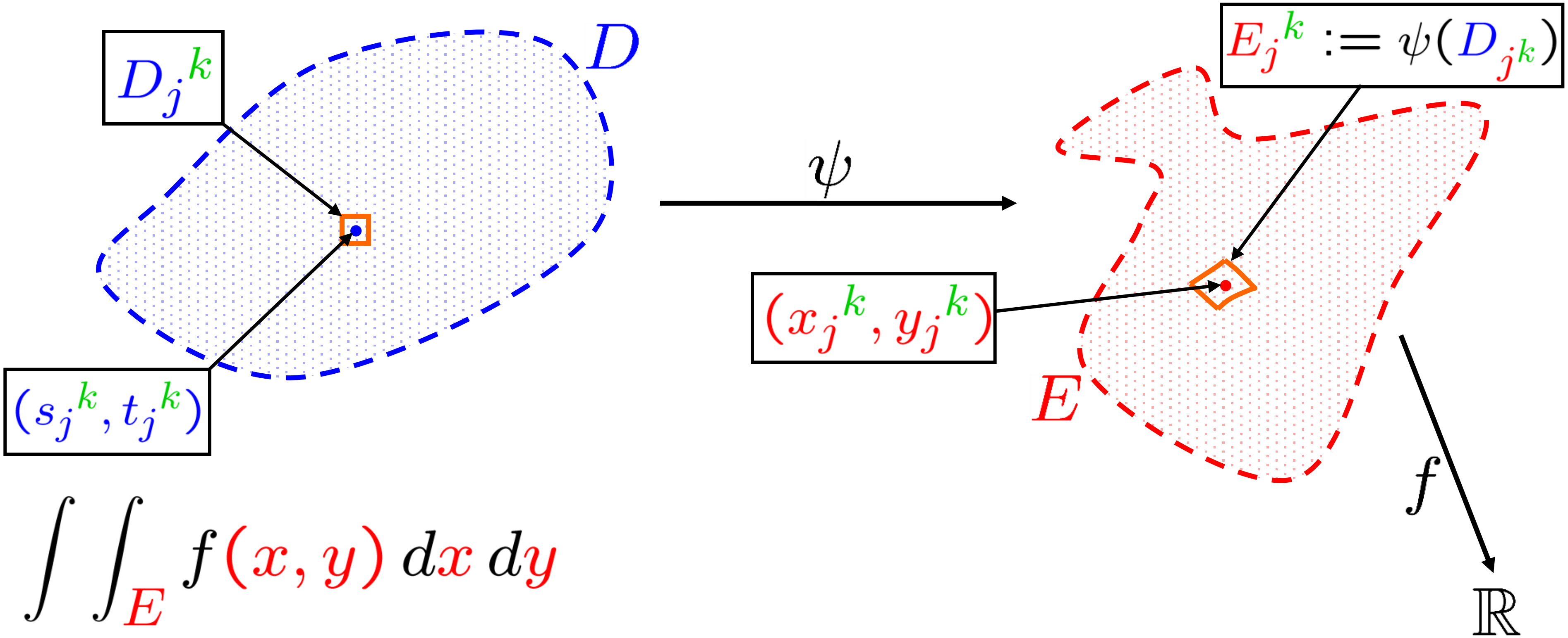
$$\approx \int\int_D [f(\psi(s, t))][|\det(\psi'(s, t))|]\, ds\, dt$$



$$\int\int_E f(x,y)\,dx\,dy$$

$$\approx \int\int_D [f(\psi(s,t))][|\det(\psi'(s,t))|]\,ds\,dt$$

$$\approx \int\int_D [f(\psi(s,t))][|\det(\psi'(s,t))|]\,ds\,dt$$

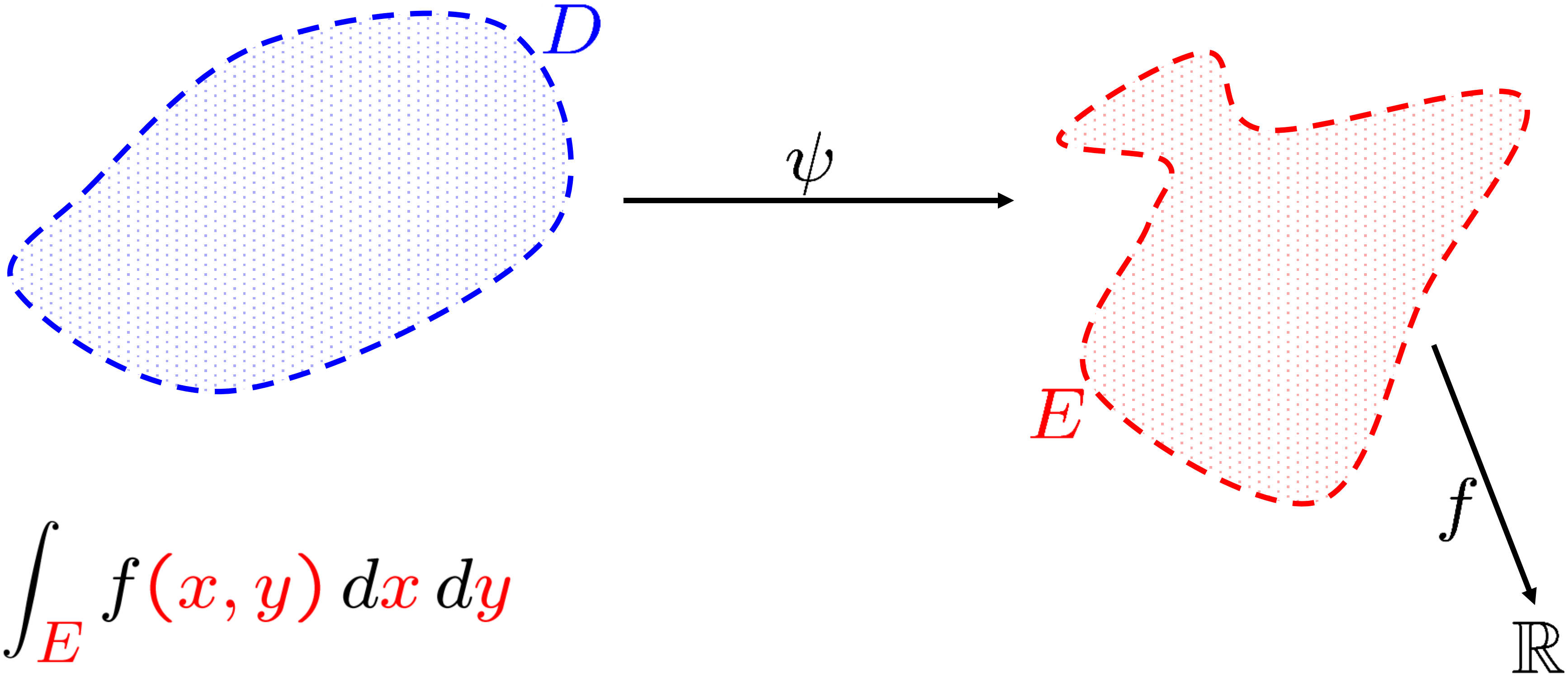


$$\int\int_E f(x,y)\,dx\,dy$$

$$\approx \int\int_D [f(\psi(s,t))][|\det(\psi'(s,t))|]\,ds\,dt$$

err $\to 0$

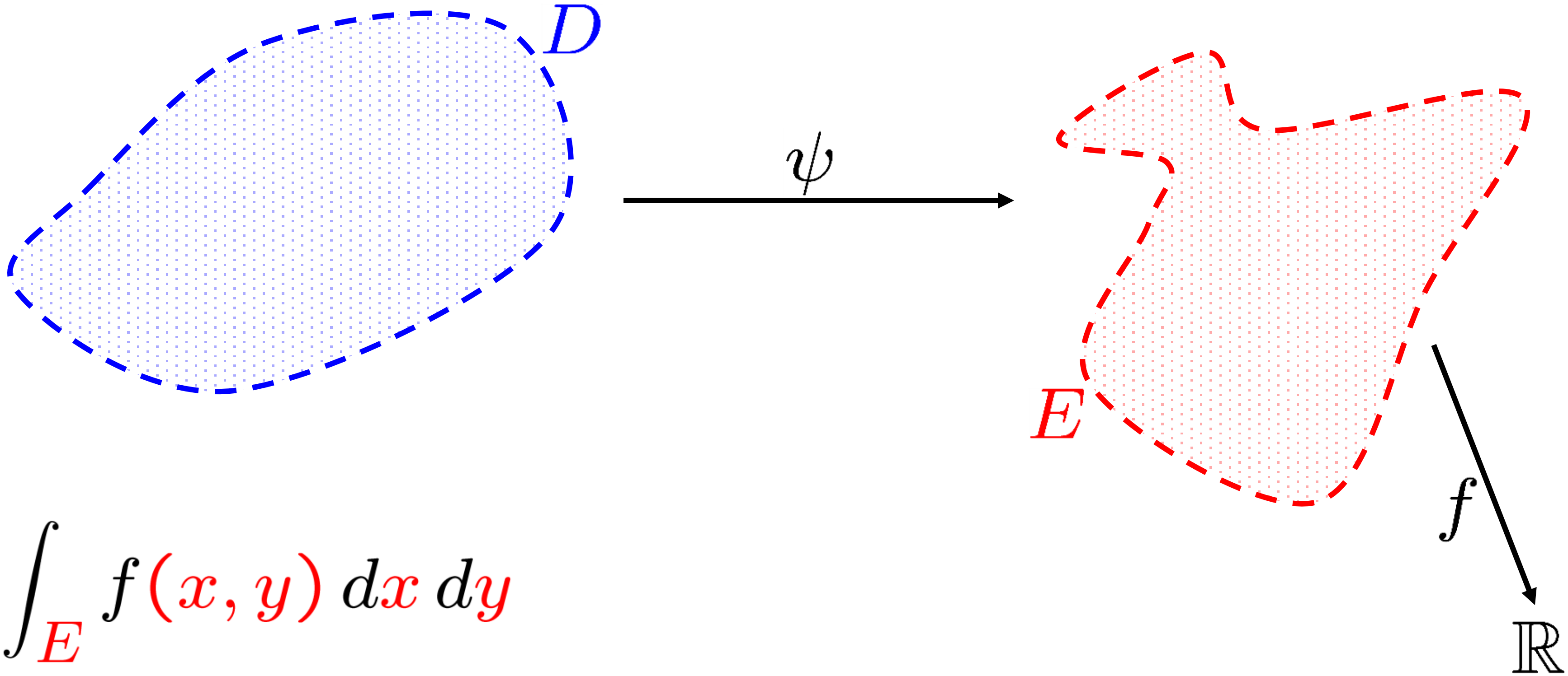
Take limit as $k \to \infty$.



$$\int\int_E f(x,y)\,dx\,dy$$

$$= \int\int_D [f(\psi(s,t))][|\det(\psi'(s,t))|]\,ds\,dt$$

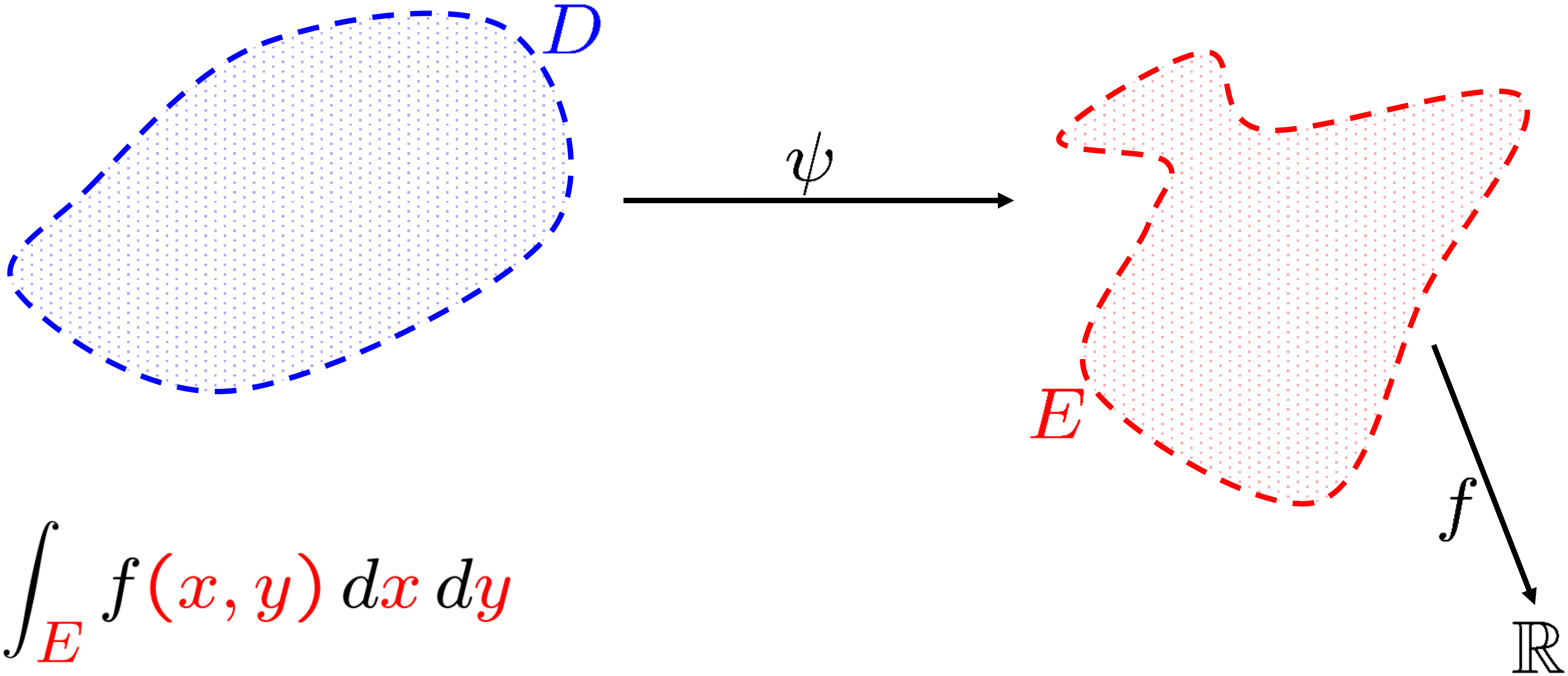
Change from (s,t) to (r,θ).



$$\int\int_{E} f(x,y)\,dx\,dy$$

$$= \int\int_{D} [f(\psi(r,\theta))][|\,\mathrm{det}(\psi'(r,\theta))|]\,dr\,d\theta$$

Change from (s,t) to (r,θ).



$$\int\int_E f(x,y)\,dx\,dy$$
$$= \int\int_D [f(\psi(r,\theta))][|\det(\psi'(r,\theta))|]\,dr\,d\theta$$

e.g.: $D := (0,\infty) \times (0, 2\pi)$

$E' := [0,\infty) \times \{0\}$ $E := \mathbb{R}^2 \backslash E'$

$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$

$f(x,y) = e^{-(x^2+y^2)/2}$

$$f(\psi(r,\theta)) = e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2} = e^{-r^2/2}$$

Colloq.: As every x is repl. by $r\cos\theta$, we write $x = r\cos\theta$. Similarly, $y = r\sin\theta$.

$$x^2 + y^2 = r^2$$

$$f(x,y) = e^{-(x^2+y^2)/2} = e^{-r^2/2}$$

$$\int\int_E f(x,y)\,dx\,dy$$

$$= \int\int_D [f(\psi(r,\theta))][|\det(\psi'(r,\theta))|]\,dr\,d\theta$$

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$E' := [0,\infty)\times\{0\}$ $\qquad E := \mathbb{R}^2\backslash E'$

$$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$f(x,y) = e^{-(x^2+y^2)/2} \qquad r^2\cos^2\theta \qquad r^2\sin^2\theta$$

$$f(\psi(r,\theta)) = e^{-(r^2\cos^2\theta + r^2\sin^2\theta)/2} = e^{-r^2/2}$$

Colloq.: As every x is repl. by $r\cos\theta$,

$$\boxed{x^2 + y^2 = r^2}$$

we write $x = r\cos\theta$.
Similarly, $y = r\sin\theta$.

$$f(x,y) = e^{-(x^2+y^2)/2} = e^{-r^2/2}$$

$$\int\int_E f(x,y)\,dx\,dy$$

$$= \int\int_D [f(\psi(r,\theta))][|\det(\psi'(r,\theta))|]\,dr\,d\theta$$

Colloq.: As $dx\,dy$ is repl. by $|\det(\psi'(r,\theta))|\,dr\,d\theta$,
we write $dx\,dy = |\det(\psi'(r,\theta))|\,dr\,d\theta$.

$$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\psi'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$dx\,dy = [|\det(\psi'(r,\theta))|]\,dr\,d\theta$$
$$= [|r\cos^2\theta - (-r\sin^2\theta)|]\,dr\,d\theta = [|r|]\,dr\,d\theta$$

$$\int\int_E f(x,y)\,dx\,dy = \int\int_D [e^{-r^2/2}][|r|]\,dr\,d\theta$$

$$f(x,y) = e^{-(x^2+y^2)/2} = e^{-r^2/2}$$

$$\int\int_E f(x,y)\,dx\,dy$$

$D := (0,\infty)\times(0,2\pi)$

$r > 0$

$$= \int\int_D [f(\psi(r,\theta))][|\det(\psi'(r,\theta))|]\,dr\,d\theta$$

Colloq.: As $dx\,dy$ is repl. by $|\det(\psi'(r,\theta))|\,dr\,d\theta$,

we write $dx\,dy = |\det(\psi'(r,\theta))|\,dr\,d\theta$.

$$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\psi'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$dx\,dy = [|\det(\psi'(r,\theta))|]\,dr\,d\theta$$
$$= [|r\cos^2\theta - (-r\sin^2\theta)|]\,dr\,d\theta = [|r|]\,dr\,d\theta$$

$$\int\int_E f(x,y)\,dx\,dy = \int\int_D [e^{-r^2/2}][\ r\ ]\,dr\,d\theta$$

$$f(x,y) = e^{-(x^2+y^2)/2} = e^{-r^2/2}$$

$$\int\int_E f(x,y)\,dx\,dy$$

$$D := (0,\infty)\times(0,2\pi)$$

$$= \int\int_D [f(\psi(r,\theta))][|\det(\psi'(r,\theta))|]\,dr\,d\theta$$

Colloq.: As $dx\,dy$ is repl. by $|\det(\psi'(r,\theta))|\,dr\,d\theta$,
we write $dx\,dy = |\det(\psi'(r,\theta))|\,dr\,d\theta$.

Colloq.: In "polar coordinates", $x = r\cos\theta$,
$y = r\sin\theta$,
and $dx\,dy = r\,dr\,d\theta$.

$$x^2 + y^2 = r^2$$

Prove:

$$0 < I := \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi}$$

Want:

$$I^2 = 2\pi$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\ dx\ dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} \quad e^{-y^2/2}\ dx\ dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2/2} \quad I \qquad dy$$

$$= I \int_{-\infty}^{\infty} e^{-y^2/2}\ dy = I^2$$

Prove:

$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi}$$

Want:

$$I^2 = 2\pi$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\ dx\ dy$$

$$= I^2$$

Prove:

$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Want:

$$I^2 = 2\pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy = I^2$$

$$\|$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\,r\,dr\,d\theta$$

$$\|$$

$$\int_0^{2\pi}\underbrace{\int_0^{\infty} e^{-s}\,ds}_{[-e^{-s}]_{s=0}^{s=\infty}}\,d\theta$$

$$x^2 + y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

$$s = r^2/2$$

$$ds = r\,dr$$

Prove:

$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Want:

$$I^2 = 2\pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy = I^2$$

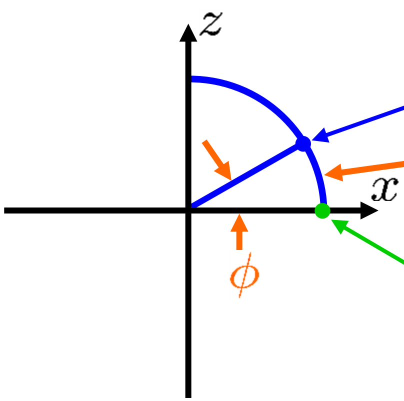
$$\|$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$

$$\|$$

$$\int_0^{2\pi}\underbrace{\int_0^{\infty} e^{-s}\, ds}_{[-e^{-s}]_{s=0}^{s=\infty}=(-0)-(-1)=1}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

QED



$$\begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

Rotate ϕ about the y-axis

$\phi \in (0, \pi/2)$

Rotate θ about the z-axis

$\theta \in (0, 2\pi)$

$r \in (0, r_0)$

Param. of $\frac{1}{2}$-ball in sph. coords... $\psi(r,\theta,\phi) =$

$$(\ r(\cos\phi)(\cos\theta)\ ,\ r(\cos\phi)(\sin\theta)\ ,\ r(\sin\phi)\ )$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos\phi)(\cos\theta) \\ r(\cos\phi)(\sin\theta) \\ r(\sin\phi) \end{bmatrix}$$

STOP

SKILL:
Use change of variables to compute the area/vol. of a well-parametrized set.