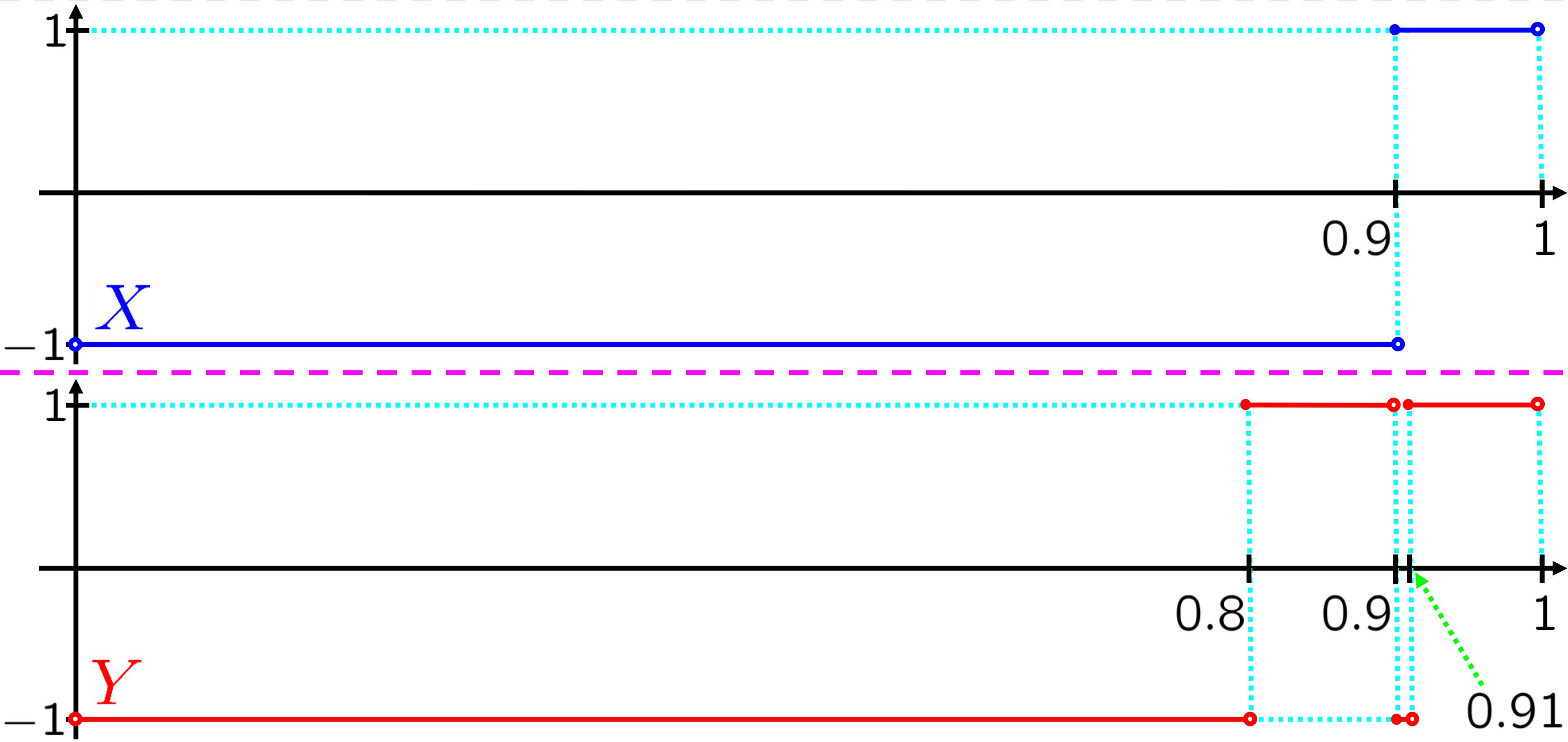
Financial Mathematics

Conditional probability, independence and the Central Limit Theorem

Conditional prob. of one event, given another



We win if $Y(\omega)$ turns out to be 1.

Tyche tells us $X(\omega)$, *then* $Y(\omega)$. In between?

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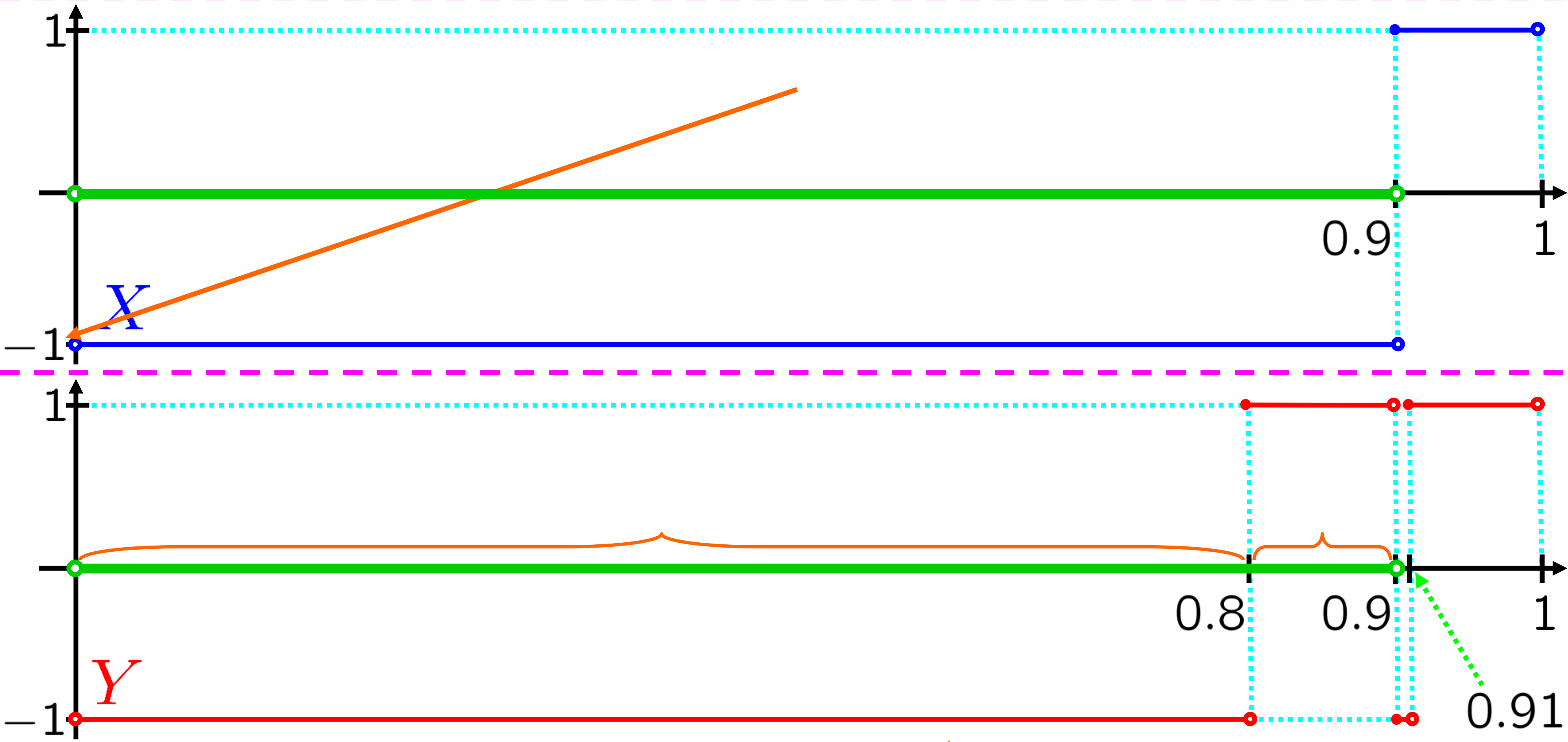

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Tyche tells us $X(\omega)$, *then* $Y(\omega)$. In between?

90% of time, $X(\omega) = -1$, & we very likely lose.

$\Pr[Y = -1 \mid X = -1] = 8/9$ 8/9 of the time

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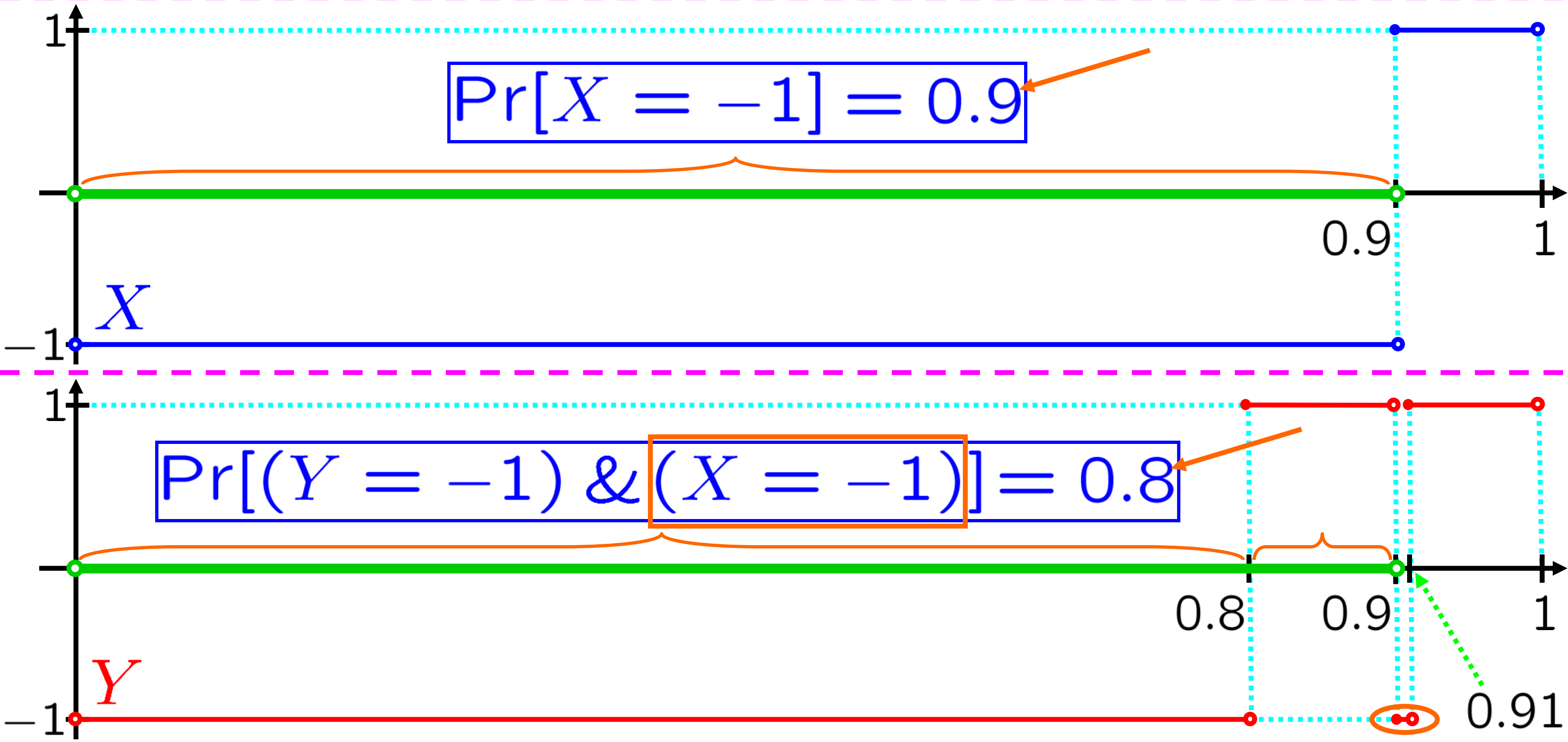

We win if $Y(\omega)$ turns out to be 1. $\frac{0.8}{0.9} = \frac{8}{9}$

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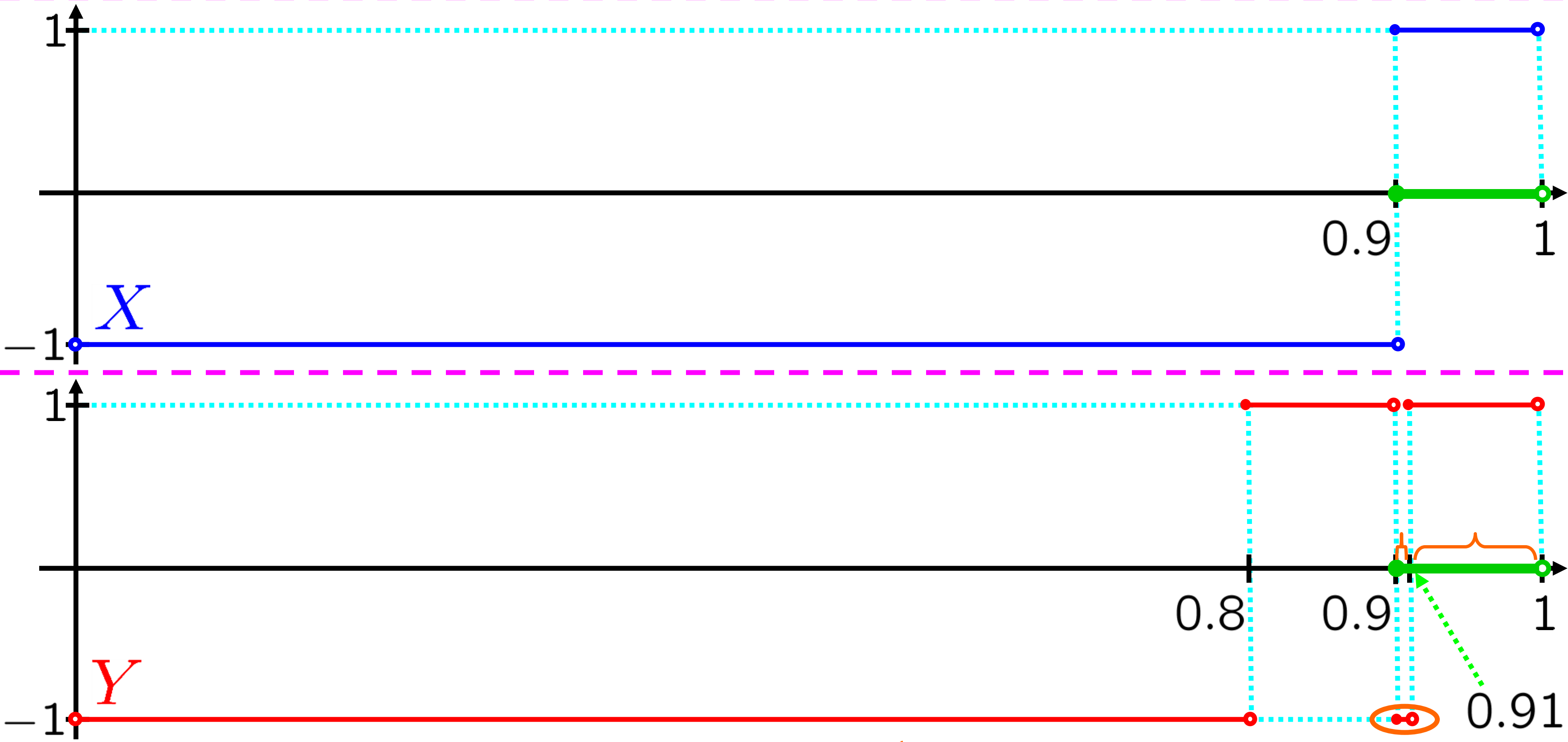


We win if $Y(\omega)$ turns out to be 1.

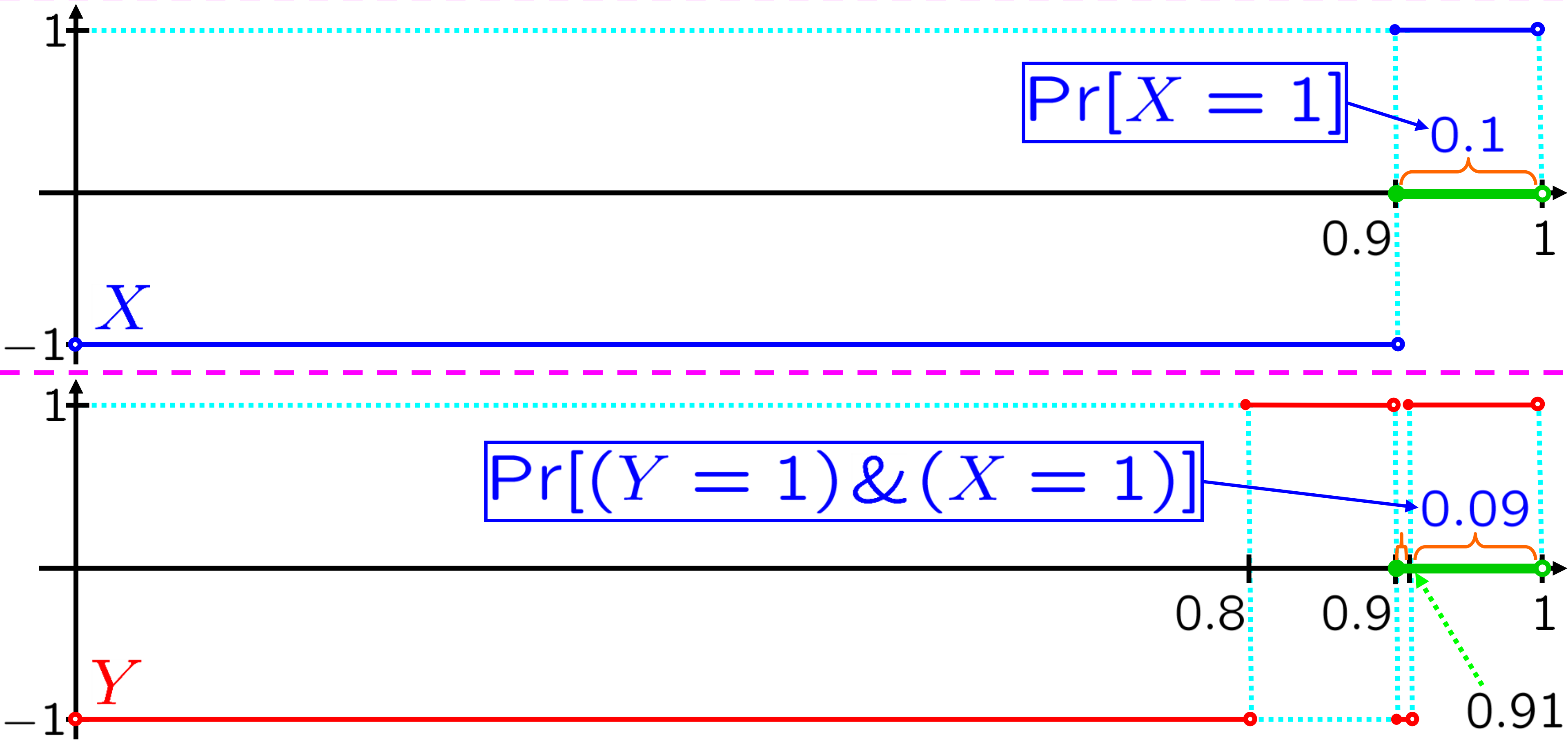
Tyche tells us $X(\omega)$, *then* $Y(\omega)$.

10% of time, $X(\omega) = 1$, & we very likely win.

$\Pr[Y = 1 | X = 1] = 9/10$

9/10 of the time

Conditional prob. of one event, given another



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Tyche tells us $X(\omega)$, *then* $Y(\omega)$.

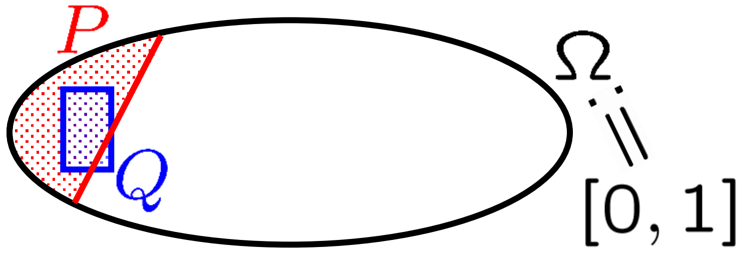
10% of time, $X(\omega) = 1$, & we very likely win.

$$\frac{0.09}{0.10} = \frac{9}{10}$$

very likely = 9/10 of the time

$\Pr[Y = 1 | X = 1] = 9/10$

Definition: The **conditional probability** of P given Q is



$$\Pr[P \mid Q] = \frac{\Pr[P \,\&\, Q]}{\Pr[Q]}$$

Warning:
Only defined when $\Pr[Q] \neq 0$.

Is P likely or unlikely?
Given that you're told Q happened,
is P likely or unlikely?

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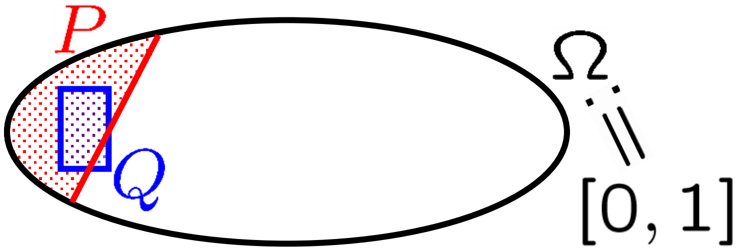

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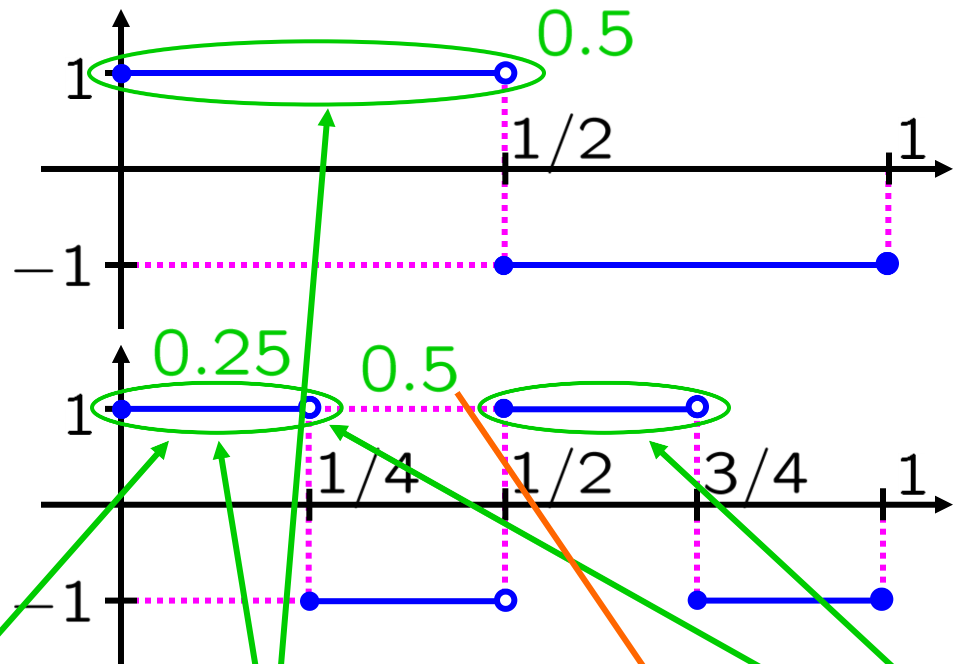
$C_1 :=$

$C_2 :=$



same distr. coin-flipping standard

Key point: Finding out $C_1 = 1$ has no influence on the prob. that $C_2 = 1$.

$$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5 = \Pr[C_2 = 1]$$

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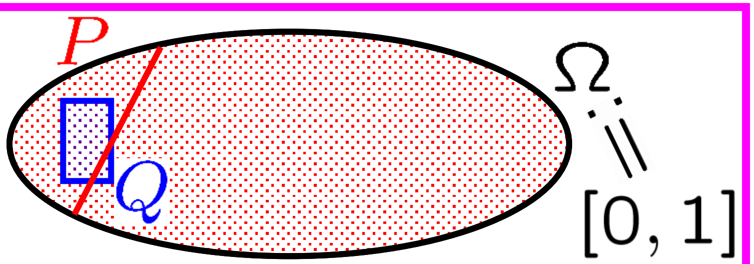

$$\Pr[P \mid Q] = \frac{\Pr[P \,\&\, Q]}{\Pr[Q]}$$

Warning: Only defined when $\Pr[Q] \neq 0$.

Definition: Assume $\Pr[Q] \neq 0$.

P & Q are **independent** (events)

if $\Pr[P \mid Q] = \Pr[P]$,

i.e.: if $\dfrac{\Pr[P \,\&\, Q]}{\Pr[Q]} = \Pr[P]$,

i.e.: if $\Pr[P \,\&\, Q] = (\Pr[P])(\Pr[Q])$.

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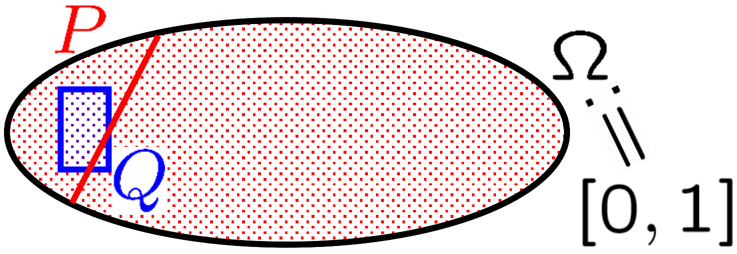

$$\boxed{\Pr[P \mid Q]} = \frac{\Pr[P \,\&\, Q]}{\Pr[Q]}$$

Warning: Only defined when $\Pr[Q] \neq 0$.

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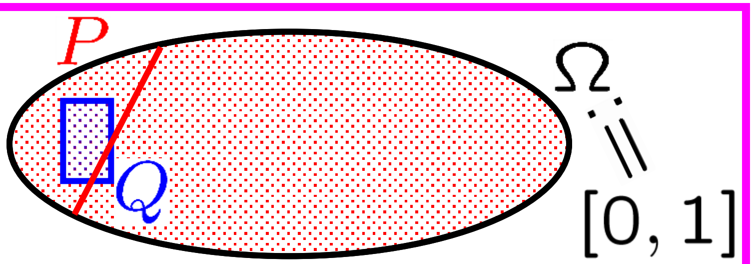

$$\boxed{\Pr[P \,|\, Q]} = \frac{\Pr[P \;\&\; Q]}{\Pr[Q]}$$

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Definition:
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"The probability of both is the product of the probabilities"

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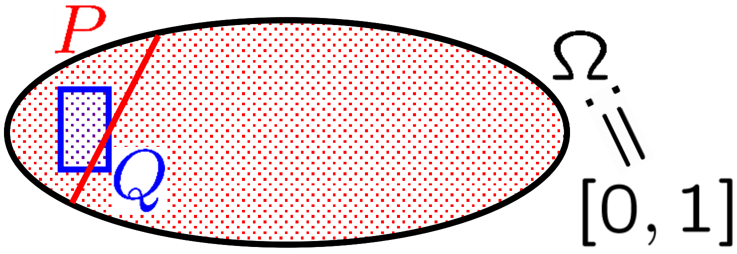

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Definition:
S & T are **independent** (PCRVs)
if, $\forall A, B \subseteq \mathbb{R}$,
$S \in A$ is independent of $T \in B$.

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$C_1 = 1$
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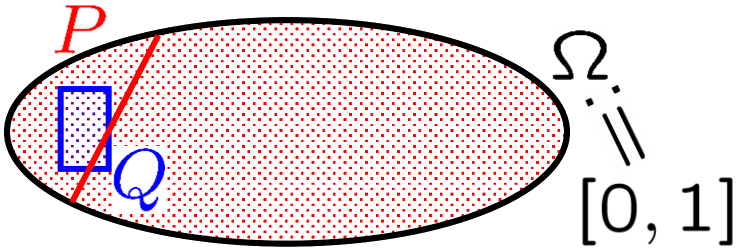

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C_1 and C_2 independent

$$\Pr[(C_2 = 1) \mid (C_1 = 1)] = \frac{0.25}{0.5} = 0.5 = \Pr[C_2 = 1]$$

these are independent

Def'ns: P, Q, R are **independent** (events) if
P, Q, R are pairwise-independent
and $\Pr[P \,\&\, Q \,\&\, R] = (\Pr[P])(\Pr[Q])(\Pr[R])$.

S, T, U are **independent** (PCRVs) if,
$\forall A, B, C \subseteq \mathbb{R}$, $S \in A$, $T \in B$ and $U \in C$ are indep.
etc., etc., etc

Definition:
P & Q are **independent** (events)
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Definition:
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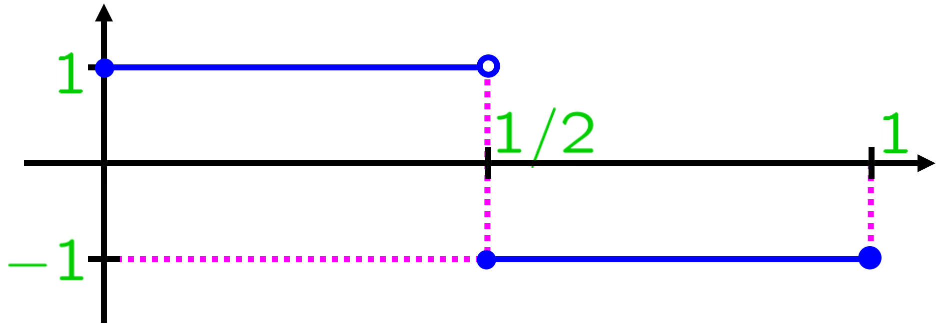
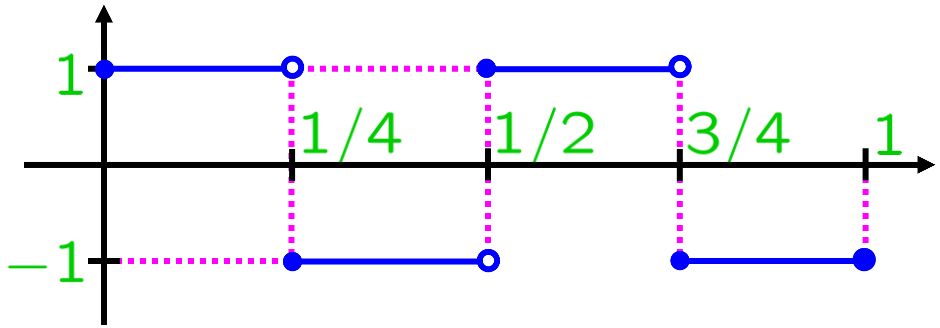
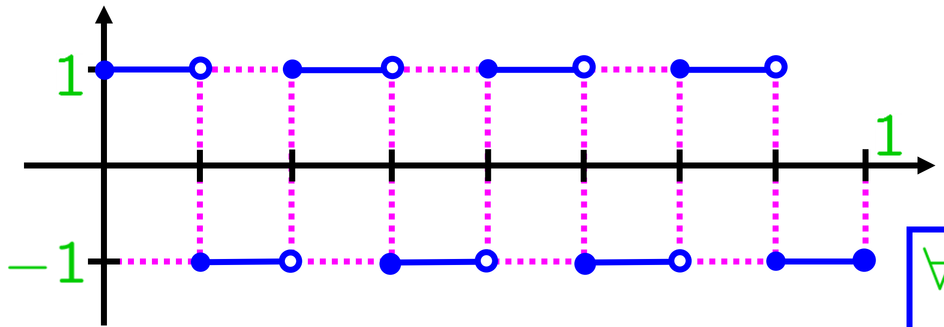


$C_2 :=$



$C_3 :=$



etc.

pairwise

$\forall$integers $j, k \geq 1$,
$j \neq k$ implies
C_j and C_k
are independent.

Exercise: Graph C_4.

Fact: $C_1, C_2, C_3, \ldots$ are pw-independent.

Stronger: Any finite set of $C_1, C_2, \ldots$
is an independent set.

Def'n: Let S and T be PCRVs.

Let $F := \{(a,b) \in \mathbb{R}^2 \mid \Pr[(S = a)\&(T = b)] > 0\}$.

The joint distribution of (S, T) associates, to each element $(a, b) \in F$, the value $\Pr[(S = a)\&(T = b)]$.

Remark: To compute the distribution of $S + T$, you need to know the JOINT distr. of (S, T).

Knowing both the distribution of S and the distribution of T is insufficient. Same for ST.

However, if S and T are independent, then their joint distribution is determined by their individual distributions, because

$$\Pr[(S = a)\&(T = b)] = (\Pr[S = a])(\Pr[T = b]).$$

All this generalizes to ≥ 2 PCRVs.

Fact: independent $\Rightarrow$ uncorrelated

Pf: Let S, T be independent PCRVs.

Want: $\text{E}[ST] = (\text{E}[S])(\text{E}[T])$

$$A := \{a \in \mathbb{R} \mid \Pr[S = a] > 0\}$$

$$B := \{b \in \mathbb{R} \mid \Pr[T = b] > 0\}$$

$$\text{E}[ST] = \sum_{a\in A}\sum_{b\in B}(\Pr[(S = a)\&(T = b)])ab$$

$$= \sum_{a\in A}\sum_{b\in B}(\Pr[S = a])(\Pr[(T = b)])ab$$

$$= \left(\sum_{a\in A}(\Pr[S = a])a\right)\left(\sum_{b\in B}(\Pr[(T = b)])b\right)$$

$$= (\text{E}[S])(\text{E}[T])$$ QED

Fact:

Let X and Y be independent PCRVs.

Then, for any functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

$f(X)$ and $g(Y)$ are independent.

The idea:

coin has $+1$ and -1 instead of H and T.

Flip a ± 1 fair coin twice.

If I tell you the first flip, you get no useful info about the second.

If I tell you $3 \times$ (the first flip) $+ 7$, you get no useful info about $5 \times$ (the second flip) $- 1$.

Fact:

Let X and Y be independent PCRVs.

Then, for any functions $f, g : \mathbb{R} \to \mathbb{R}$,

$f(X)$ and $g(Y)$ are independent.

Proof: Given $S, T \subseteq \mathbb{R}$.

Want: $\text{Pr}[(f(X) \in S) \;\&\; (g(Y) \in T)]$

$= (\text{Pr}[f(X) \in S])(\text{Pr}[g(Y) \in T])$

$$\text{Pr}[(f(X) \in S) \;\&\; (g(Y) \in T)]$$
$$= \text{Pr}[(X \in f^{-1}(S)) \;\&\; (Y \in g^{-1}(T))]$$
$$= (\text{Pr}[X \in f^{-1}(S)])(\text{Pr}[Y \in g^{-1}(T)])$$
$$= (\text{Pr}[f(X) \in S])(\text{Pr}[g(Y) \in T])$$

QED

Fact:

Let X and Y be independent PCRVs.

Then, for any functions $f, g : \mathbb{R} \to \mathbb{R}$,

$f(X)$ and $g(Y)$ are independent.

Fact: independent $\Rightarrow$ uncorrelated

Restatement:

Let A and B be independent PCRVs.

Then $\mathsf{E}[AB] = (\mathsf{E}[A])(\mathsf{E}[B])$.

Corollary:

Let X and Y be independent PCRVs.

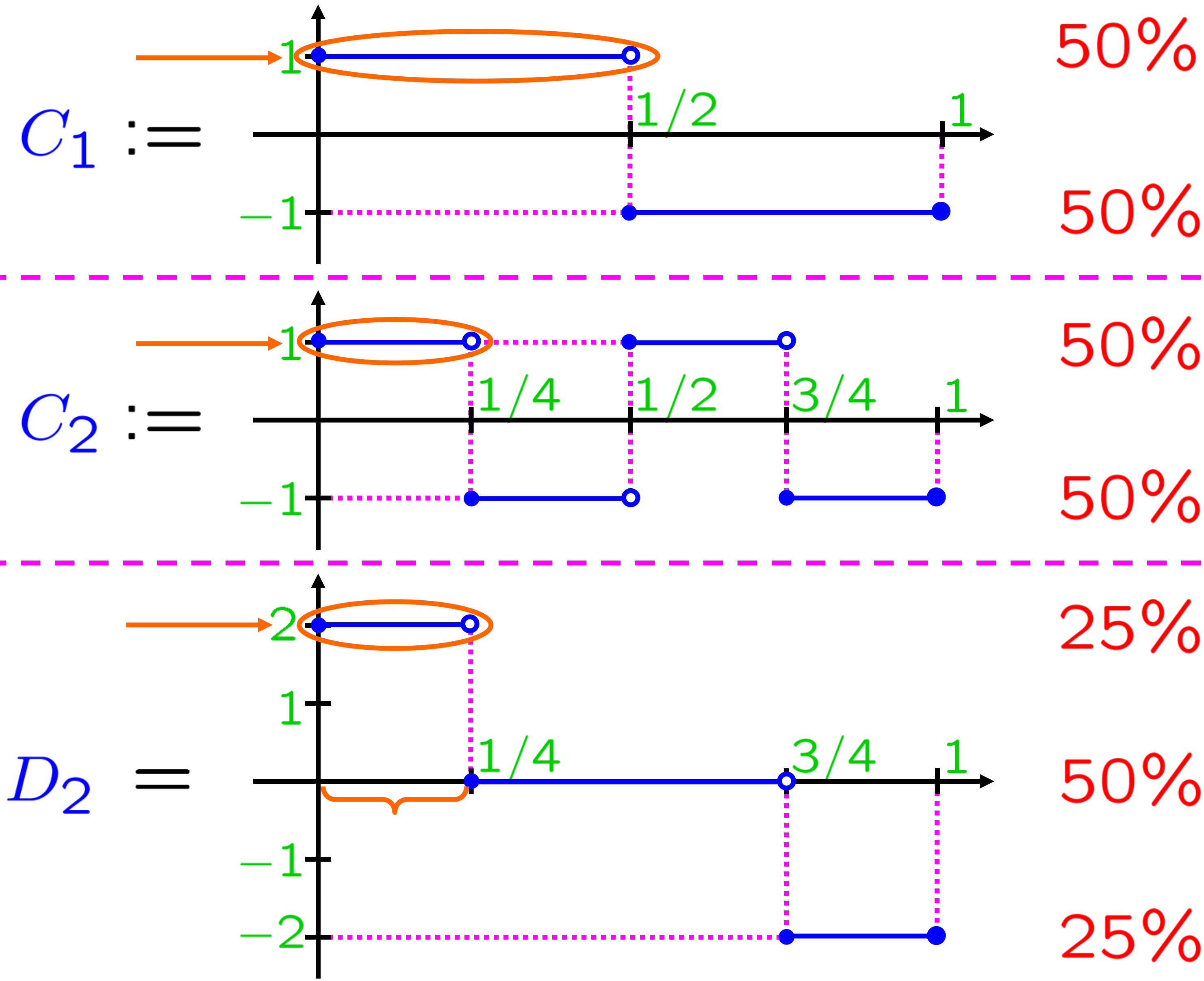
Then, for any functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$\mathsf{E}[(f(X))(g(Y))] = (\mathsf{E}[f(X)])(\mathsf{E}[g(Y)]).$$

Rmk: Converse is true, too. pf omitted

Definition: $\forall$integers $n > 0$, $D_n := C_1 + \cdots + C_n$

models (#heads) − (#tails) after n flips of a fair coin



models (#heads) − (#tails) after n flips of a fair coin

Definition: $\forall$integers $n > 0$,

$$D_n ::= C_1 + \cdots + C_n$$

Fact: independent $\Rightarrow$ uncorrelated,

i.e., S, T independent $\Rightarrow$

$$\text{Var}[S + T] = \text{Var}[S] + \text{Var}[T].$$

$C_1, \ldots, C_n$ are all standard (*i.e.*, mean 0 variance 1)

$$\text{E}[D_n] = (\text{E}[C_1]) + \cdots + (\text{E}[C_n])$$
$$= 0 + \cdots + 0 = 0$$

$$\text{Var}[D_n] = (\text{Var}[C_1]) + \cdots + (\text{Var}[C_n])$$
$$= 1 + \cdots + 1 = n$$

$$\text{E}\left[\frac{D_n}{\sqrt{n}}\right] = 0 \quad \text{and} \quad \text{Var}\left[\frac{D_n}{\sqrt{n}}\right] = 1,$$

i.e., $\frac{D_n}{\sqrt{n}}$ is standard.

Definition: $\forall$integers $n > 0$, models (#heads) − (#tails) after n flips of a fair coin

$$D_n := C_1 + \cdots + C_n$$

Preview of the Central Limit Theorem:

$$\frac{D_n}{\sqrt{n}} \to Z \quad \boxed{\text{in distribution}}, \quad \text{as } n \to \infty.$$

standard normal random variable

Definition?

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Definition: $\forall$integers $n > 0$, models (#heads) − (#tails) after n flips of a fair coin

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Preview of the Central Limit Theorem:

$$\frac{D_n}{\sqrt{n}} \to Z$$ in distribution, as $n \to \infty$.

standard normal random variable

Definition?

$\forall$test functions ψ,

$$\mathsf{E}\left[\psi\left(\frac{D_n}{\sqrt{n}}\right)\right] \to \mathsf{E}[\psi(Z)]$$

$$\| $$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}]\,dx$$

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Relatively easy: "test function" = "continuous, compactly supported function"

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Relatively easy: "test function" = "continuous, compactly supported function"

Harder to prove: "test function" = "continuous, exponentially-bounded function"

f **exponentially bounded** means:

$\exists A, B > 0$ s.t. $\forall x \in \mathbb{R}$, $|f(x)| \leq Ae^{B|x|}$

Definition: $\forall$integers $n > 0$, $D_n := C_1 + \cdots + C_n$ — models (#heads) − (#tails) after n flips of a fair coin

Preview of the Central Limit Theorem:

$\forall$continuous, exponentially-bounded ψ,

$$\mathrm{E}\left[\psi\left(\frac{D_n}{\sqrt{n}}\right)\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}]\, dx$$

Exercise: Compute $\lim_{n\to\infty} \mathrm{E}\left[\left(e^{D_n/\sqrt{n}} - 7\right)_+\right]$.

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$$\mathrm{E}\left[\psi\left(\frac{D_n}{\sqrt{n}}\right)\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\psi(x)][e^{-x^2/2}]\,dx$$

Solution: $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(e^x - 7)_+][e^{-x^2/2}]\,dx = \cdots$

exp-bdd $\longrightarrow \psi(x) = (e^x - 7)_+$

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$$\mathsf{E}\left[\psi\left(\frac{D_n}{\sqrt{n}}\right)\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(x)][e^{-x^2/2}]\, dx$$

e^{R_n}

Hint: $\psi(x) := e^{ax+b}$

Def'n: $\forall X$, the **augmented expectation** of X is defined by $\mathsf{AE}[X] := (\mathsf{E}[X]) + \frac{1}{2}(\mathsf{Var}[X])$.

"asymptotically normal"

Fact: Fix $a, b \in \mathbb{R}$. Let $R_n := a\left(\frac{D_n}{\sqrt{n}}\right) + b$.

"E *almost* asymptotically commutes with $e^\bullet$"

Then $\lim_{n\to\infty} \mathsf{E}[e^{R_n}] = \lim_{n\to\infty} e^{\mathsf{AE}[R_n]}$.

Pf: $\lim_{n\to\infty} \mathsf{E}[e^{R_n}] \overset{\text{CLT}}{\underset{\text{exercise}}{=}} e^b e^{a^2/2} \overset{\text{CLT}}{\underset{\text{exercise}}{=}} \lim_{n\to\infty} e^{\mathsf{AE}[R_n]}$.

models (#heads) − (#tails) after n flips of a fair coin

Definition: $\forall$integers $n > 0$,

$$D_n := C_1 + \cdots + C_n$$

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R_n^2

Hint: $\psi(x) := (ax+b)^2$

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Pf: $\lim_{n\to\infty} \mathrm{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{\underset{\text{exercise}}{=}} \lim_{n\to\infty} e^{\mathrm{AE}[R_n]}$. QED "normal" "standard normal"

Fact: Fix $a, b \in \mathbb{R}$. Let $R := aZ + b$.

"E *almost* commutes with $e^{\bullet}$ . . .

Then $\mathsf{E}[e^R] \doteq e^{\mathsf{AE}[R]}$.

. . . but we need to go from the expectation to the augmented expectation"

Next subtopic: mean/var of summand from mean/var of iid sum

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Then $\lim_{n\to\infty} \mathsf{E}[e^{R_n}] \doteq \lim_{n\to\infty} e^{\mathsf{AE}[R_n]}$.

Pf: $\lim_{n\to\infty} \mathsf{E}[e^{R_n}] \overset{\text{CLT}}{=} e^b e^{a^2/2} \overset{\text{CLT}}{=} \lim_{n\to\infty} e^{\mathsf{AE}[R_n]}$. QED

independent, identically distributed

Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathrm{E}[X_1] = \cdots = \mathrm{E}[X_n]$$

$$\sigma := \mathrm{SD}[X_1] = \cdots = \mathrm{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathrm{E}[S] = 0.225181512$,

$\mathrm{SD}[S] = 0.158877565$. Find μ and σ.

Def'n: $\forall X$, the **augmented expectation of** X is defined by $\mathrm{AE}[X] := (\mathrm{E}[X]) + \frac{1}{2}(\mathrm{Var}[X])$.

"asymptotically normal"

Fact: Fix $a, b \in \mathbb{R}$. Let $R_n := a\left(\frac{D_n}{\sqrt{n}}\right) + b$.

"E *almost* asymptotically commutes with $e^{\bullet}$"

Then $\lim\limits_{n\to\infty} \mathrm{E}[e^{R_n}] = \lim\limits_{n\to\infty} e^{\mathrm{AE}[R_n]}$.

Pf: $\lim\limits_{n\to\infty} \mathrm{E}[e^{R_n}] \overset{\mathrm{CLT}}{=} e^b e^{a^2/2} \overset{\mathrm{CLT}}{=} \lim\limits_{n\to\infty} e^{\mathrm{AE}[R_n]}$. QED

independent, identically distributed

Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathsf{E}[X_1] = \cdots = \mathsf{E}[X_n]$$

$$\sigma := \mathsf{SD}[X_1] = \cdots = \mathsf{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathsf{E}[S] = 0.225181512$,

$\mathsf{SD}[S] = 0.158877565$. Find μ and σ.

Solution:

$$0.225181512$$

$$\|$$

$$\mathsf{E}[S] = \mathsf{E}[X_1] + \cdots + \mathsf{E}[X_n]$$

$$= n\mu = (12)\mu,$$

so $\mu = 0.225181512/12$

independent, identically distributed

Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathrm{E}[X_1] = \cdots = \mathrm{E}[X_n]$$

$$\sigma := \mathrm{SD}[X_1] = \cdots = \mathrm{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathrm{E}[S] = 0.225181512$,

$\mathrm{SD}[S] = 0.158877565$. Find μ and σ.

Solution: $\mu = 0.225181512/12$

$$\mathrm{Var}[S] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n]$$

$$\mu = 0.225181512/12$$

independent, identically distributed

Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathrm{E}[X_1] = \cdots = \mathrm{E}[X_n]$$

$$\sigma := \mathrm{SD}[X_1] = \cdots = \mathrm{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathrm{E}[S] = 0.225181512$,

$\mathrm{SD}[S] = 0.158877565$. Find μ and σ.

Solution: $\mu = 0.225181512/12$

$$(0.158877565)^2$$

$$\|$$

$$\mathrm{Var}[S] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n]$$

$$= n\sigma^2 = (12)\sigma^2,$$

so $\sigma^2 = (0.158877565)^2/12$

so $\sigma = 0.158877565/\sqrt{12}$

independent, identically distributed

Exercise: Let $n := 12$. Assume $X_1, \ldots, X_n$ iid.

$$\mu := \mathrm{E}[X_1] = \cdots = \mathrm{E}[X_n]$$

$$\sigma := \mathrm{SD}[X_1] = \cdots = \mathrm{SD}[X_n]$$

Let $S := X_1 + \cdots + X_n$.

Assume $\mathrm{E}[S] = 0.225181512$,

$\mathrm{SD}[S] = 0.158877565$. Find μ and σ.

Solution: $\mu = 0.225181512/12$

$\sigma = 0.158877565/\sqrt{12}$ ■

$$\sigma = 0.158877565/\sqrt{12}$$

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Let $S := X_1 + \cdots + X_n$.

Assume $\mathrm{E}[S] = 0.225181512$,

$\mathrm{SD}[S] = 0.158877565$. Find μ and σ.

Solution: $\mu = 0.225181512/12$

$\sigma = 0.158877565/\sqrt{12}$ ■

Mean and variance are cut by a factor of 12.
Standard deviation is cut by a factor of $\sqrt{12}$.

Conversely, on adding n *uncorrelated* PCRVs, SD increases by a factor of $\sqrt{n}$, NOT n.
A portfolio of *uncorrelated* assets is better. . .

Let's explore this. . .

$$\text{Var}[A+B] = (\text{Var}[A]) + (\text{Var}[B]) + 2(\text{Cov}[A,B])$$

$$\text{E}[A+B] \;=\; (\text{E}[A]) \;+\; (\text{E}[B])$$

Say A and B are prices, one month from now, of two financial assets.

If $\text{E}[A]$ is large, then A becomes attractive.
If $\text{E}[B]$ is large, then B becomes attractive.

If $\text{Var}[A]$ is small, then A becomes attractive.
If $\text{Var}[B]$ is small, then B becomes attractive.

If $\text{Cov}[A,B]$ is small or, even better, negative, then the portfolio of A *and* B becomes attractive.

Cauchy-Schwarz:

$$-\sqrt{\mathsf{Var}[A]}\sqrt{\mathsf{Var}[B]} \leq \mathsf{Cov}[A,B] \leq \sqrt{\mathsf{Var}[A]}\sqrt{\mathsf{Var}[B]}$$

Definition:

A and B are **perfectly correlated** if

$$\mathsf{Cov}[A,B] = \sqrt{\mathsf{Var}[A]}\sqrt{\mathsf{Var}[B]}$$

Definition:

A and B are **perfectly anti-correlated** if

$$-\sqrt{\mathsf{Var}[A]}\sqrt{\mathsf{Var}[B]} = \mathsf{Cov}[A,B]$$

Cauchy-Schwarz:

$$-\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]} \leq \text{Cov}[A,B] \leq \sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}$$

$\forall$non-deterministic PCRVs A, B,

$$\text{Corr}[A,B] := \frac{\text{Cov}[A,B]}{\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}}$$

the correlation of A and B

$$-1 \leq \text{Corr}[A,B] \leq 1$$

Suppose A and B are non-determinstic PCRVs.

$\text{Corr}[A,B] = 1$ if and only if A and B are perfectly correlated.

$\text{Corr}[A,B] = 0$ if and only if A and B are uncorrelated.

$\text{Corr}[A,B] = -1$ if and only if A and B are perfectly anti-correlated.

Definition:

A and B are **positively correlated** if

$$\text{Cov}[A, B] > 0$$

(equiv., for non-det. A and B: $\text{Corr}[A, B] > 0$).

Definition:

A and B are **negatively correlated** if

$$\text{Cov}[A, B] < 0$$

(equiv., for non-det. A and B: $\text{Corr}[A, B] < 0$).

Definition:

A and B are **uncorrelated** if

$$\text{Cov}[A, B] = 0$$

(equiv., for non-det. A and B: $\text{Corr}[A, B] = 0$).

If A and B are uncorrelated,
or, even better, negatively correlated
then the portfolio of A *and* B
becomes attractive.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\forall \text{PCRVs } X,\ \boxed{\text{SD}[X]} := \sqrt{\text{Var}[X]}$$

$\text{Var}[2X] = 4(\text{Var}[X])$ $\text{SD}[2X] = 2(\text{SD}[X])$	$\text{Var}[cX] = c^2(\text{Var}[X])$ $\text{SD}[cX] = \lvert c\rvert(\text{SD}[X])$

Intuition: Variance measures risk, but standard deviation measures risk better, because doubling the position really ought only to double the risk.

Definition: **Standard deviation** $:= \sqrt{\text{Variance}}$

$$\forall \text{PCRVs } X, \boxed{\text{SD}[X]} := \sqrt{\text{Var}[X]}$$

$$\text{Var}[A+B] = (\text{Var}[A]) + (\text{Var}[B]) + 2(\text{Cov}[A,B])$$

$$\text{SD}[A+B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A,B])}$$

$$\text{Corr}[A, B] := \frac{\text{Cov}[A, B]}{\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}}$$

$$\text{SD}[A + B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A, B])}$$

$$\text{SD}[A + B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A, B])}$$

$$\text{Corr}[A,B] := \frac{\text{Cov}[A,B]}{\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}}$$

$$\text{SD}[A+B] = \sqrt{(\text{SD}[A])^2 + (\text{SD}[B])^2 + 2(\text{Cov}[A,B])}$$

Assume $\text{Corr}[A,B] = 1$. MULTIPLY BY $\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}$

Then $\text{Cov}[A,B] = 1\sqrt{\text{Var}[A]}\sqrt{\text{Var}[B]}$
$= (\text{SD}[A])(\text{SD}[B])$.

Then $\text{SD}[A+B] = \sqrt{[(\text{SD}[A]) + (\text{SD}[B])]^2}$
$= (\text{SD}[A]) + (\text{SD}[B])$.

For perfectly correlated PCRVs,
standard deviations add.

$$\mathrm{Corr}[A,B] := \frac{\mathrm{Cov}[A,B]}{\sqrt{\mathrm{Var}[A]}\sqrt{\mathrm{Var}[B]}}$$

$$\mathrm{Var}[A+B] = (\mathrm{Var}[A]) + (\mathrm{Var}[B]) + 2(\mathrm{Cov}[A,B])$$

Assume $\mathrm{Corr}[A,B] = 0$.

Then $\mathrm{Cov}[A,B] = 0 \cdot \sqrt{\mathrm{Var}[A]}\sqrt{\mathrm{Var}[B]} = 0$.

Then $\mathrm{Var}[A+B] = (\mathrm{Var}[A]) + (\mathrm{Var}[B])$

For uncorrelated PCRVs, variances add.

For perfectly correlated PCRVs, standard deviations add.