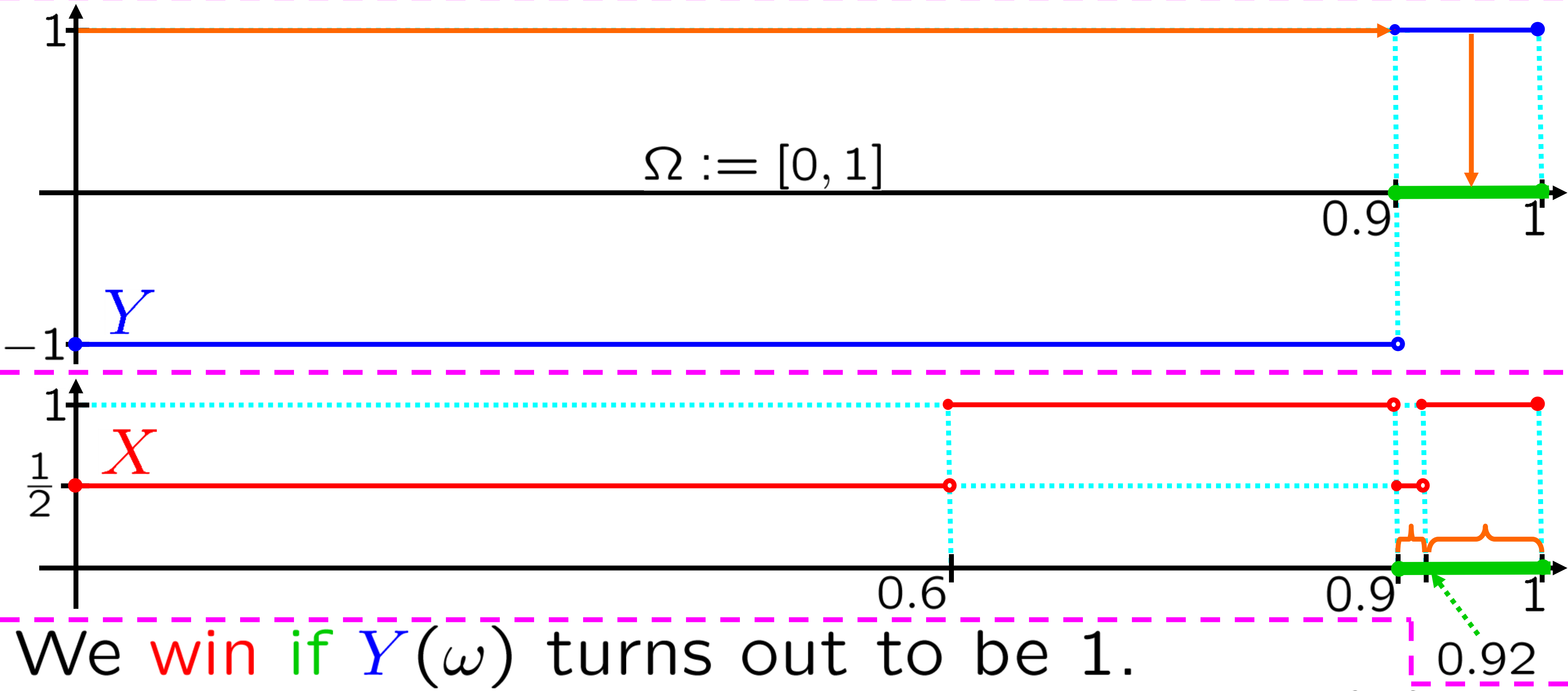
Financial Mathematics

Conditional expectation

Conditional exp. of a PCRV, given an event



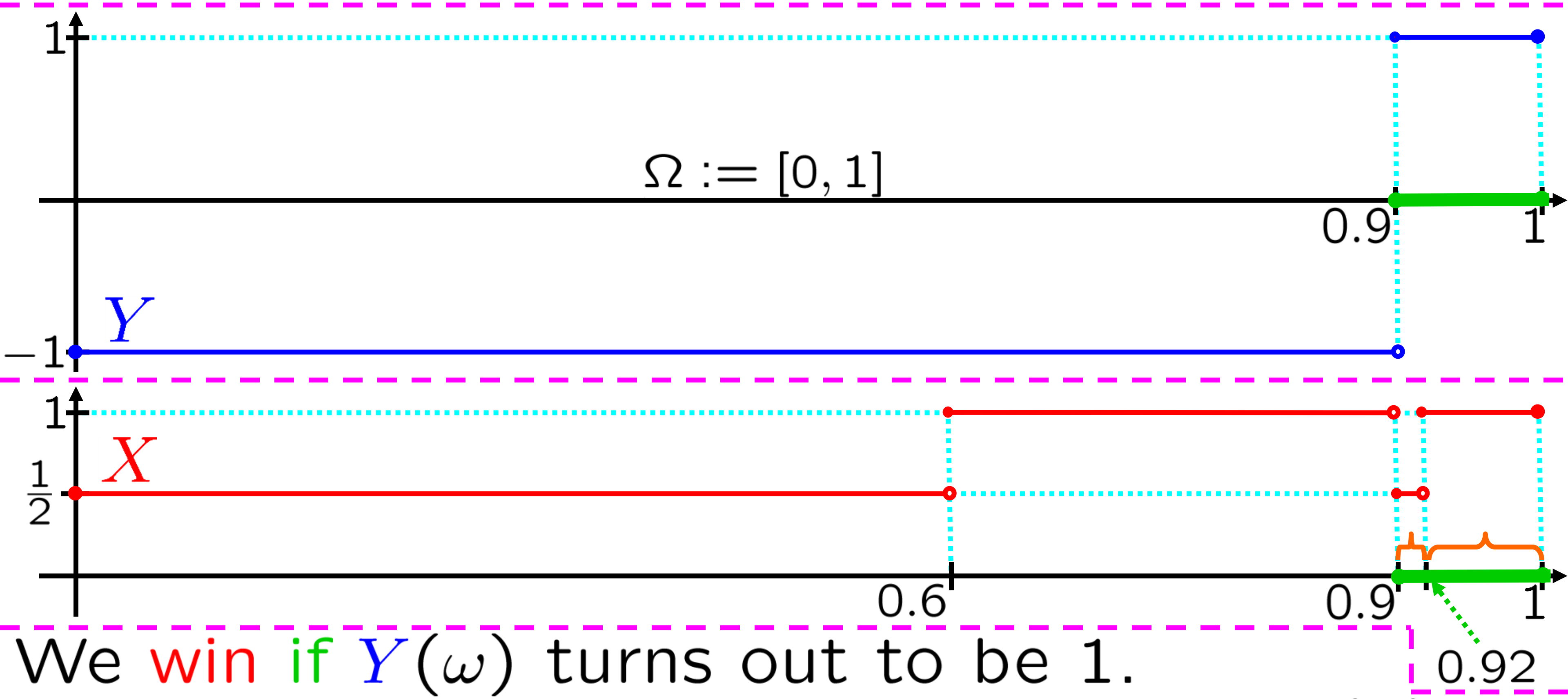
We win if $Y(\omega)$ turns out to be 1.
If we win, then the amount we win is $X(\omega)$.
During the 10% of the time that we win,

20% of that time, we win 1/2

and 80% of that time, we win 1.

Let's start with an example. . .

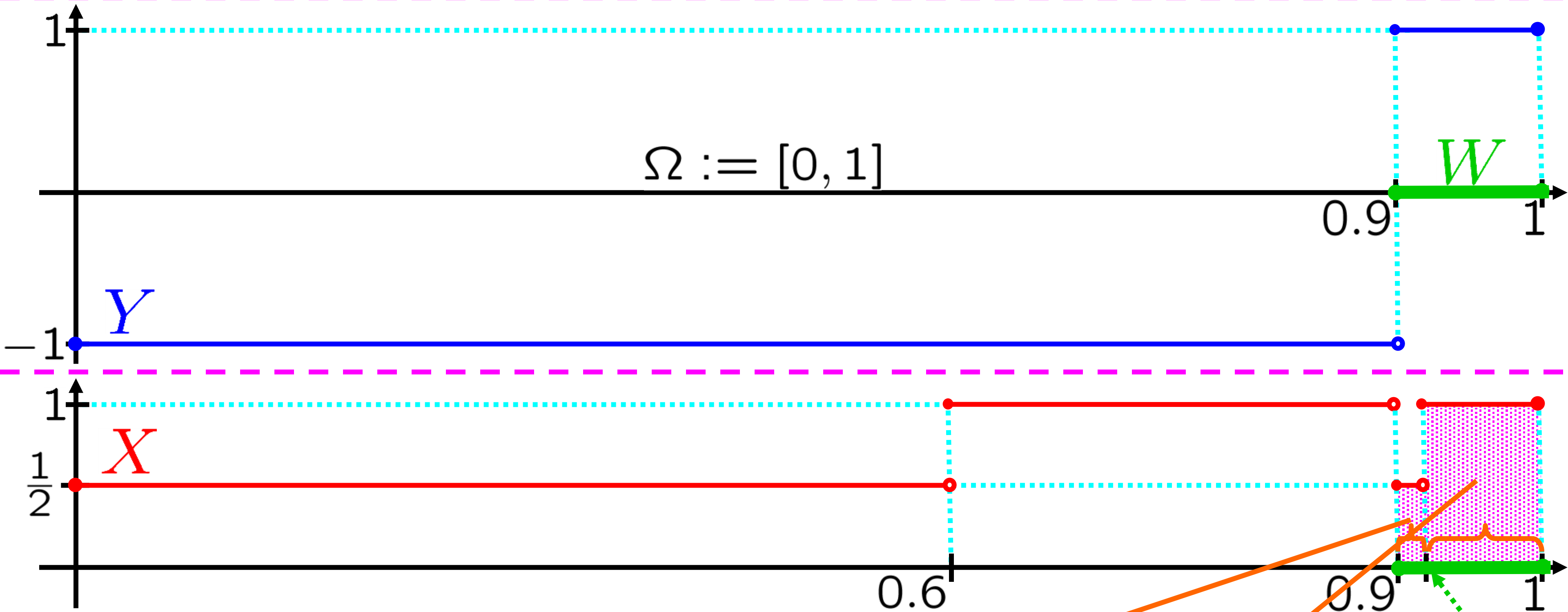
Conditional exp. of a PCRV, given an event



We win if $Y(\omega)$ turns out to be 1.
If we win, then the amount we win is $X(\omega)$.
During the 10% of the time that we win,
20% of that time, we win 1/2
and 80% of that time, we win 1.
Expected winnings, *given* that we win:

$$(20\%)(1/2) + (80\%)(1) = 0.9$$

Conditional exp. of a PCRV, given an event



0.92

$$W := \{\omega \in \Omega \mid Y(\omega) = 1\}$$

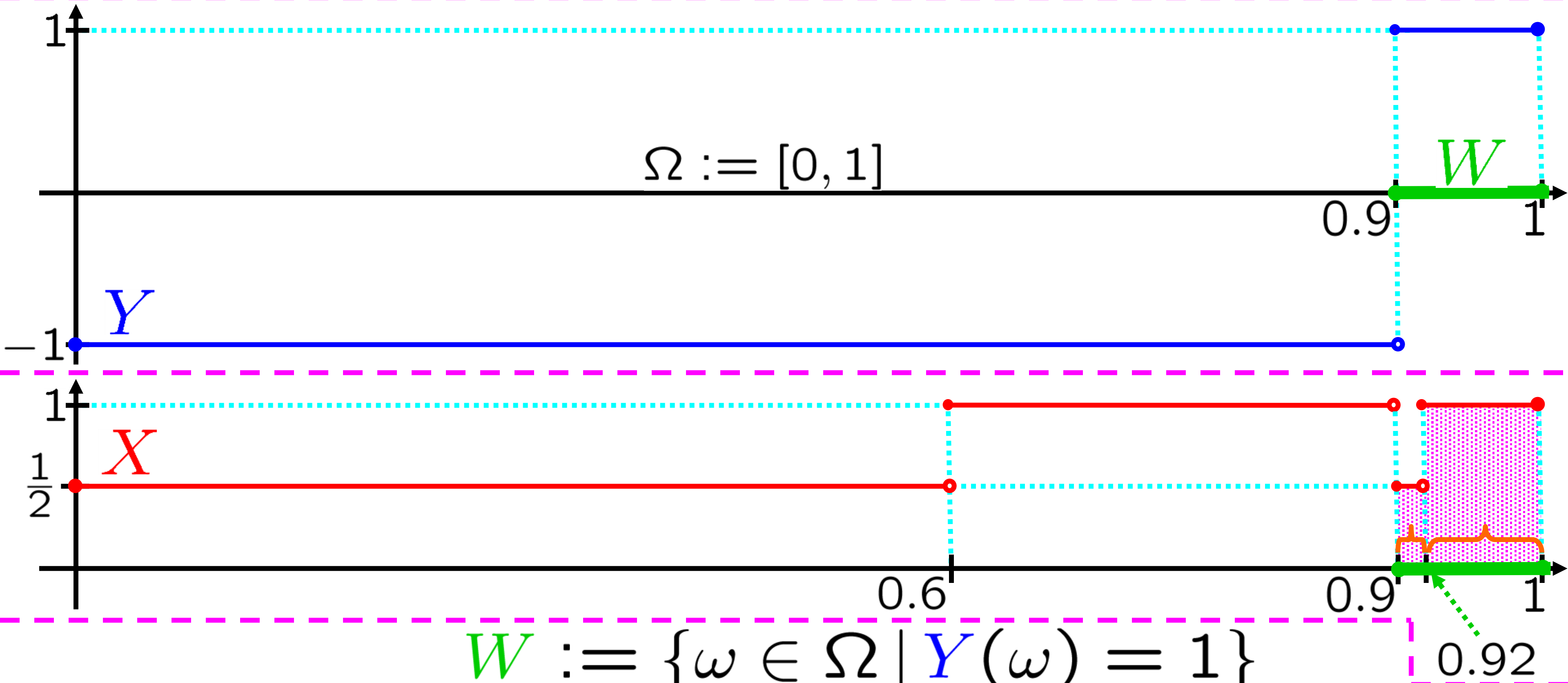
$$\int_W X\, d\lambda = (0.02)(1/2) + (0.08)(1) = 0.09$$

$$\mathrm{E}[X|W] := \frac{1}{|W|}\int_W X\, d\lambda = 0.9$$

Expected winnings, *given* that we win:

$(20\%)(1/2) + (80\%)(1) = 0.9$

Conditional exp. of a PCRV, given an event



$$W := \{\omega \in \Omega \,|\, Y(\omega) = 1\}$$

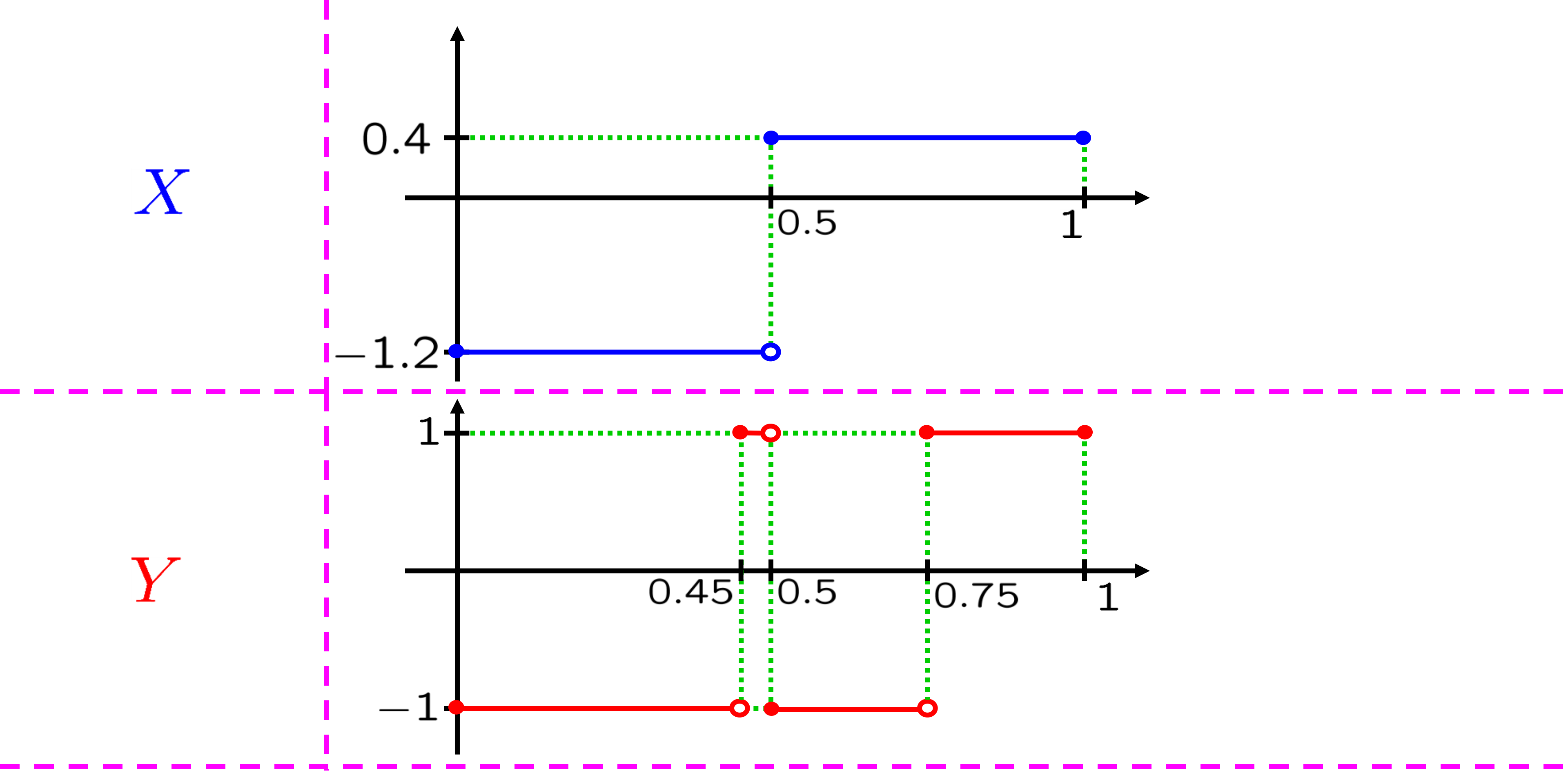
the **conditional expectation** of X given W

$$\mathrm{E}[X|W] := \frac{1}{|W|}\int_W X\, d\lambda = 0.9$$

Expected winnings, *given* that we win:

$$(20\%)(1/2) + (80\%)(1) = 0.9$$

Conditional exp. of a PCRV, given another PCRV

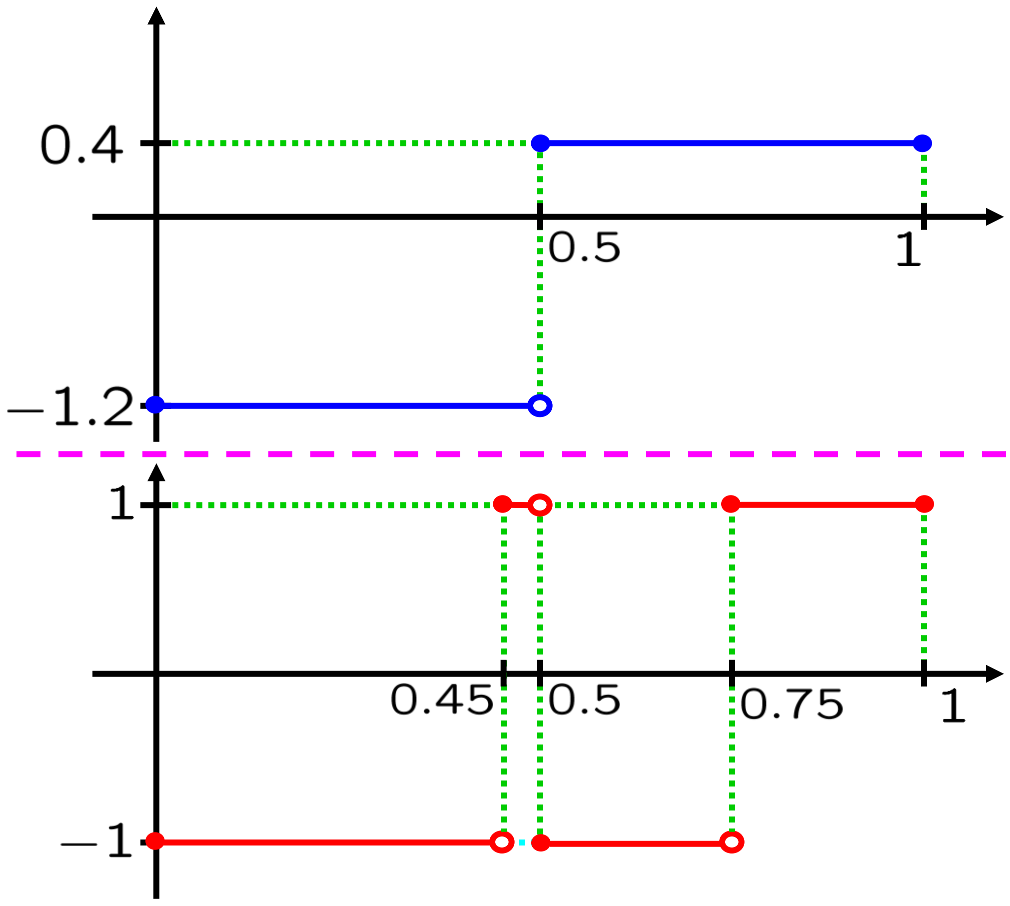


$$E[Y|X=-1.2] = (-0.45+0.05)/(0.5) = -0.8$$

$$E[Y|X=0.4] = (-0.25+0.25)/(0.5) = 0$$

X

Y



$E[Y|X]$

$E[Y|X=-1.2]$ −1.2]

$\|$

-0.8

$\|$

-0.8

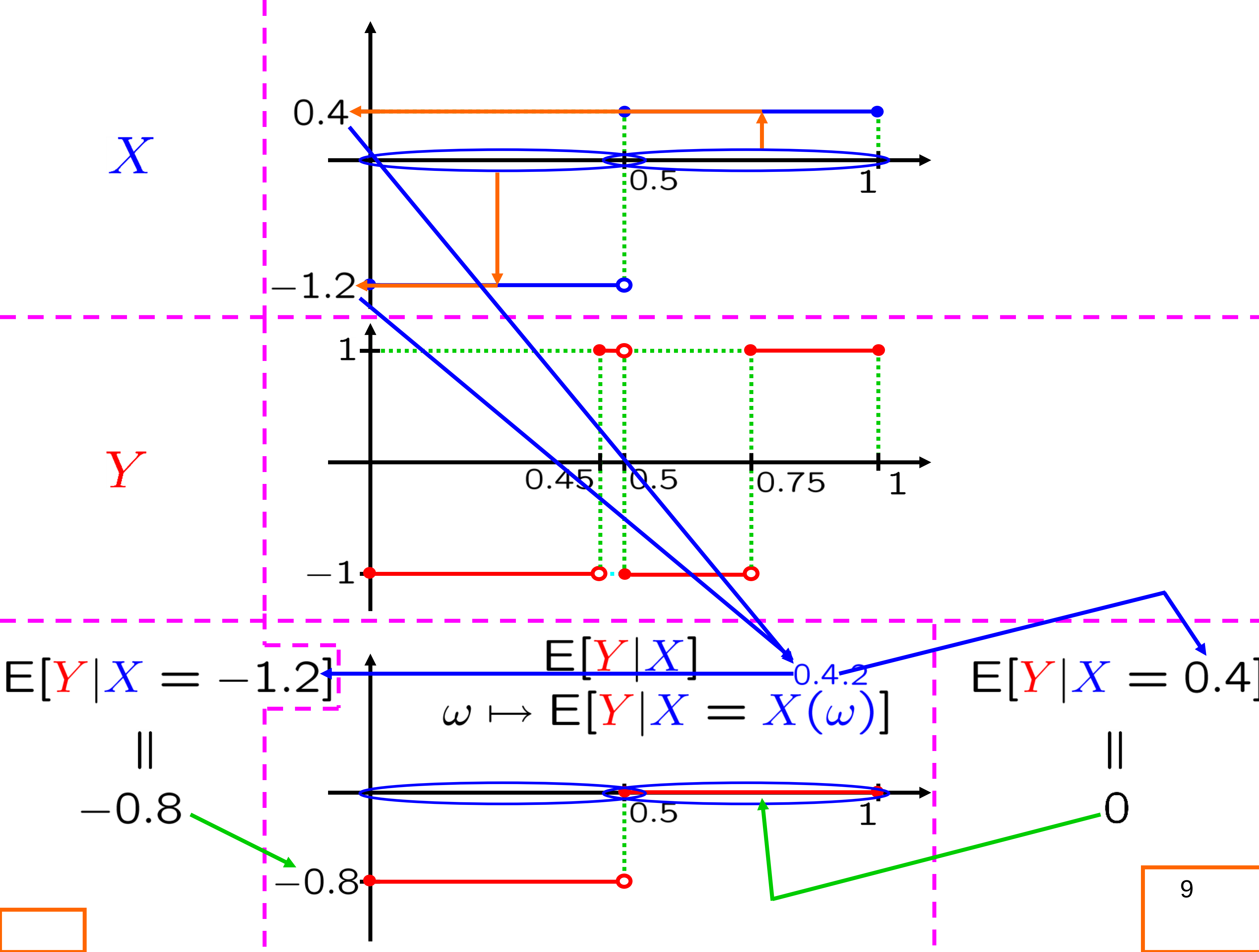
$E[Y|X$ $E[Y|X=0.4]$

$\|$

0

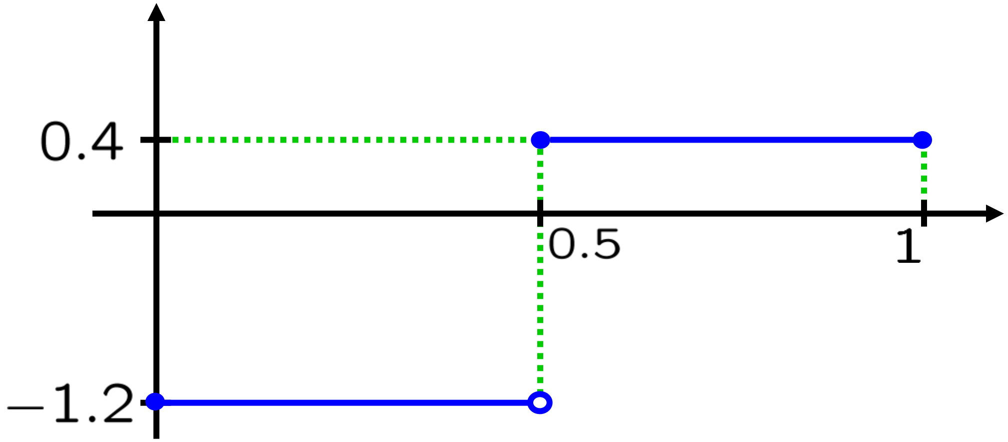
$\|$

0

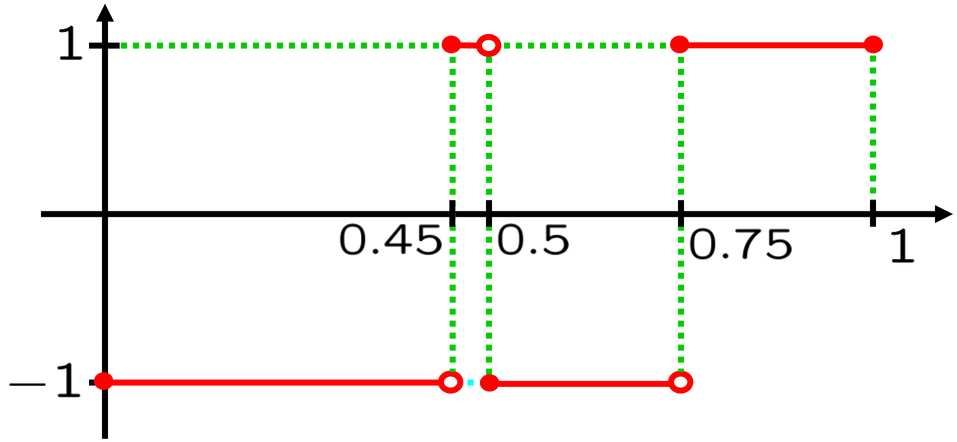


X
Y
0.4
−1.2
0.5
1
0.45
0.75
−1
$E[Y|X]$
$E[Y|X = -1.2]$
$\omega \mapsto E[Y|X = X(\omega)]$
$E[Y|X = 0.4]$
||
−0.8
0
0.4.2

X



Y



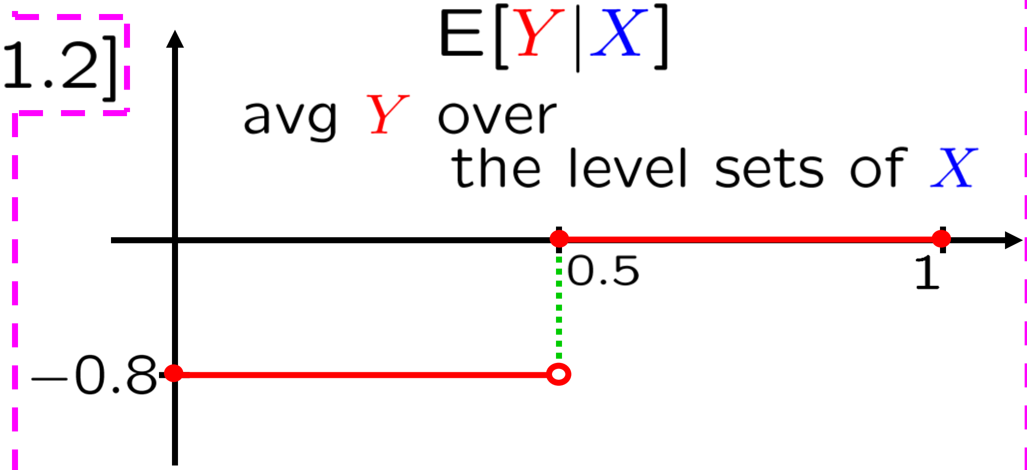
$E[Y|X = -1.2]$

||

-0.8

$E[Y|X]$

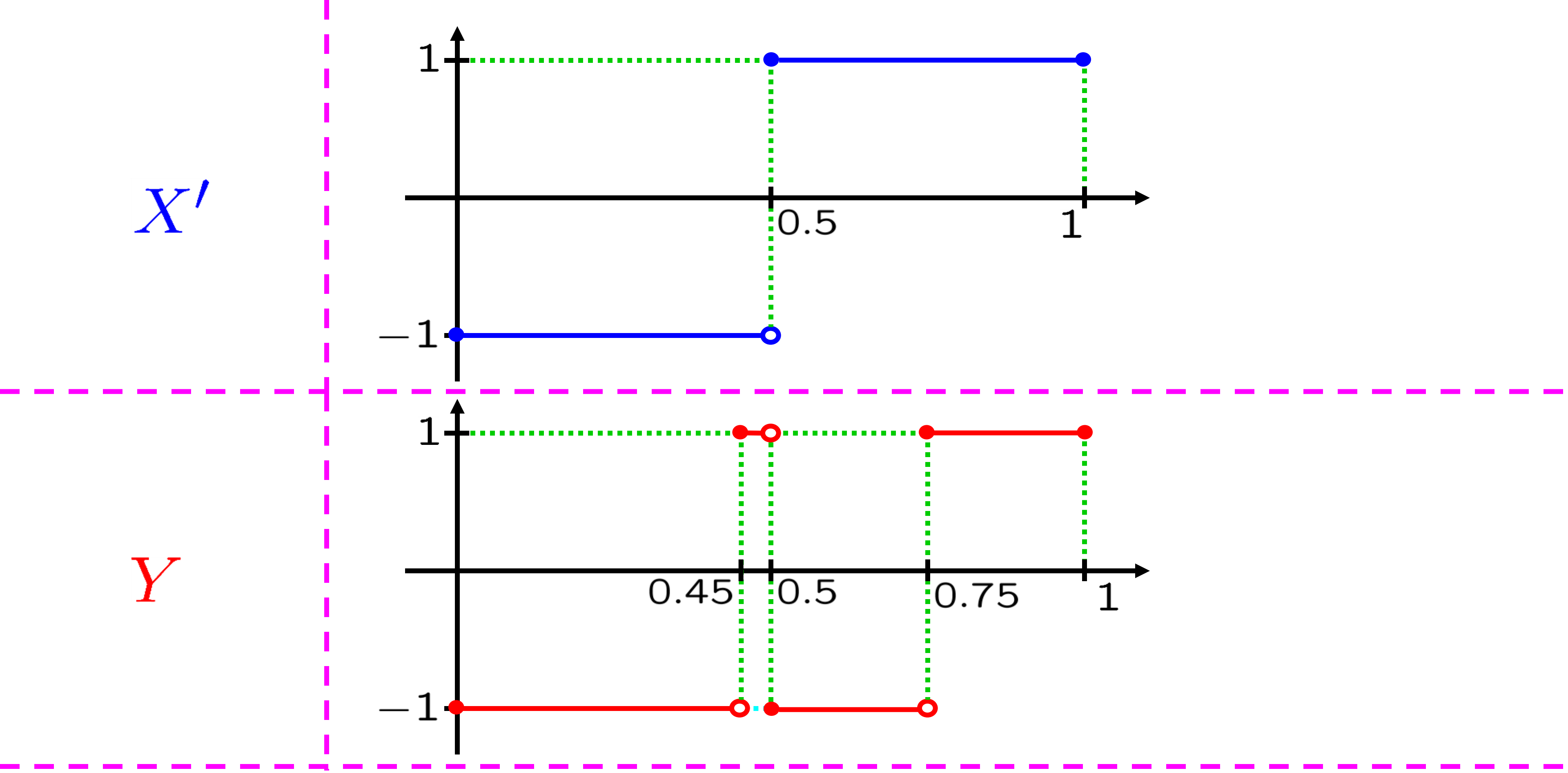
avg Y over the level sets of X



$E[Y|X = 0.4]$

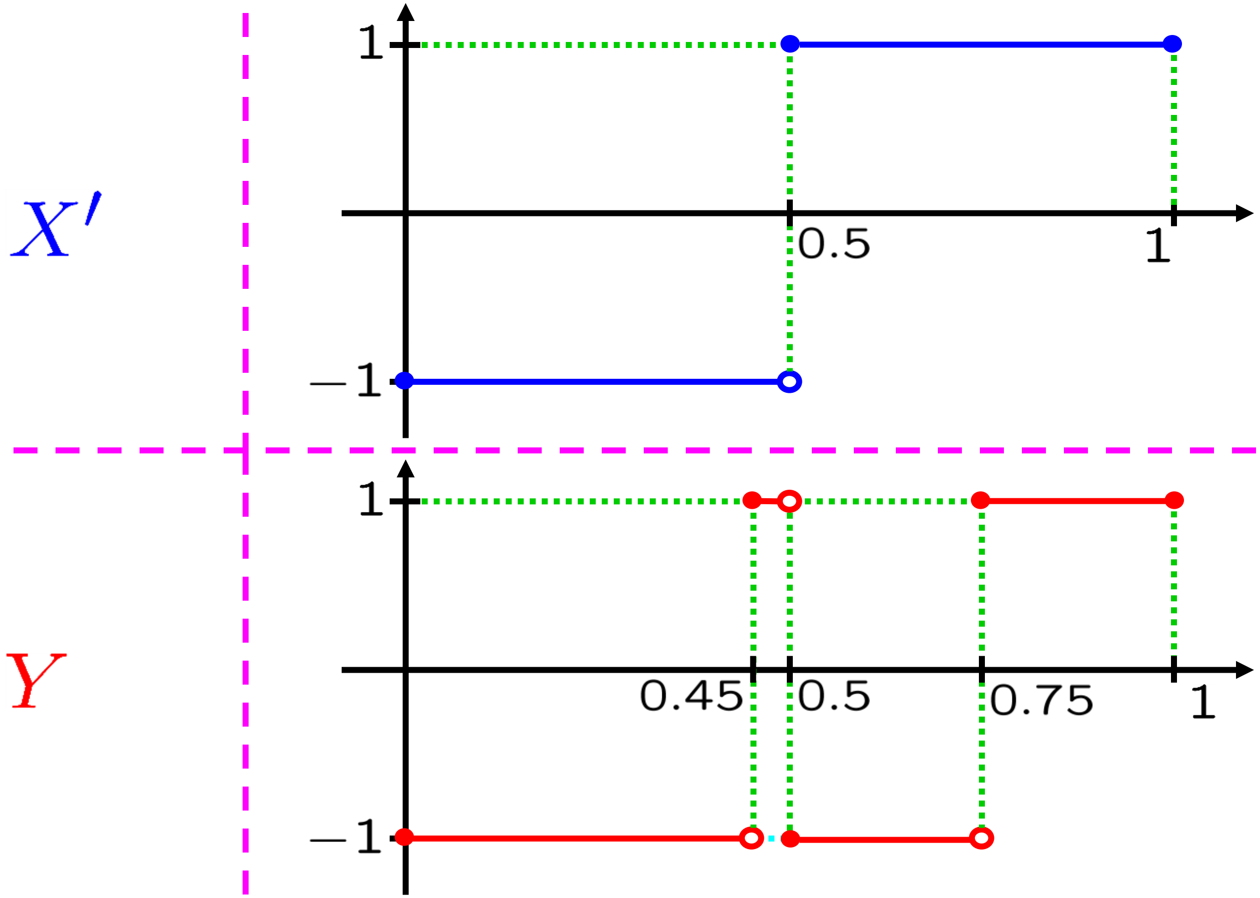
||

0



$$E[Y|X' = -1] = (-0.45 + 0.05)/(0.5) = -0.8$$

$$E[Y|X' = 1] = (-0.25 + 0.25)/(0.5) = 0$$



$\mathrm{E}[Y|X']$

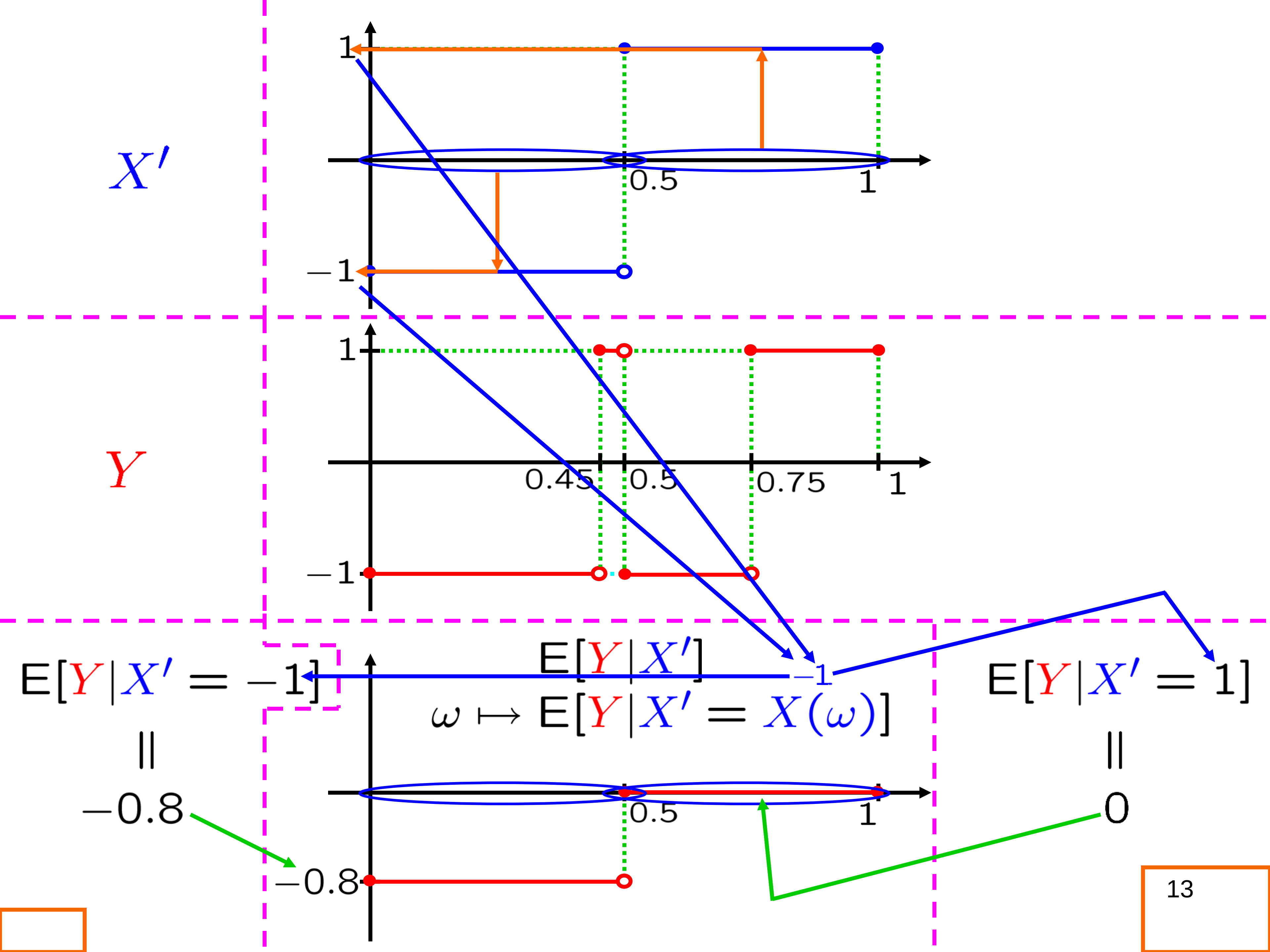
$\mathrm{E}[Y|X\mathrm{E}[Y|X' = -1]$ $\qquad$ $\mathrm{E}[Y|X' = 1] = 1]$

$\|$ $\qquad$ $\|$

-0.8 $\qquad$ 0

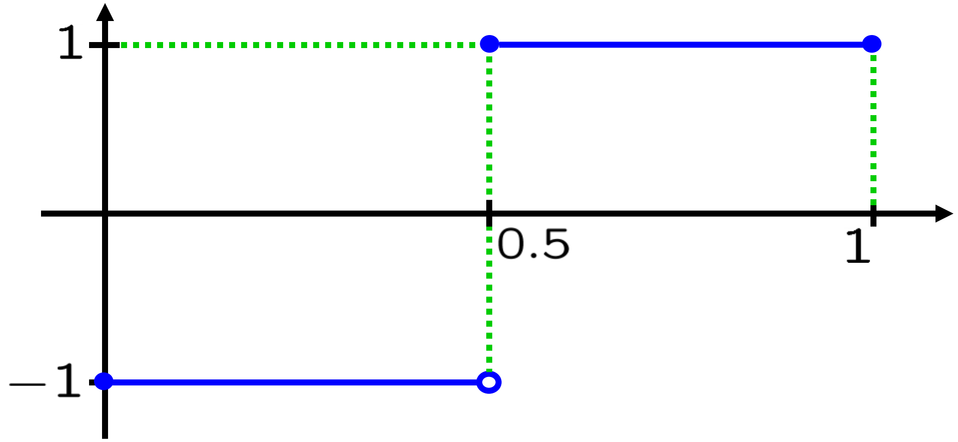
$\|$ $\qquad$ $\|$

-0.8 $\qquad$ 0

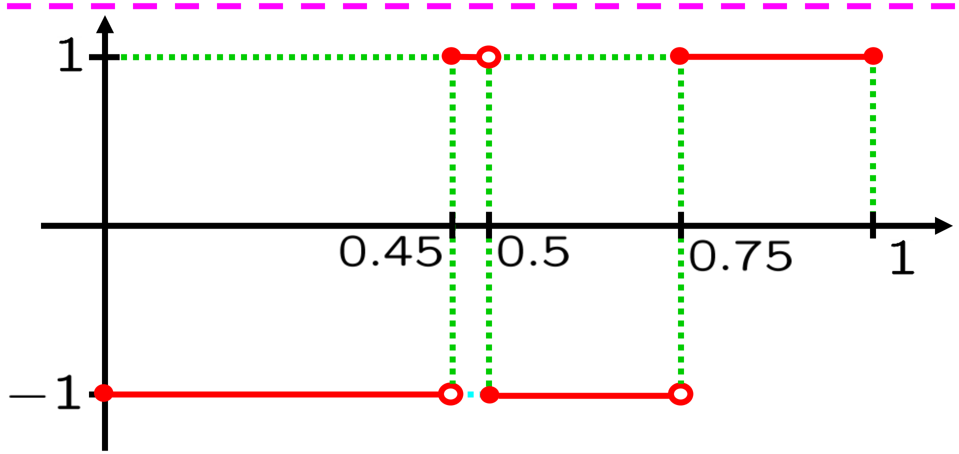


X'
Y
1
-1
0.5
1
0.45
0.5
0.75
1
-1
$E[Y|X'=-1]$
||
-0.8
$E[Y|X']$
$\omega \mapsto E[Y|X'=X(\omega)]$
-1
$E[Y|X'=1]$
||
0
0.5
1
-0.8

X'



Y



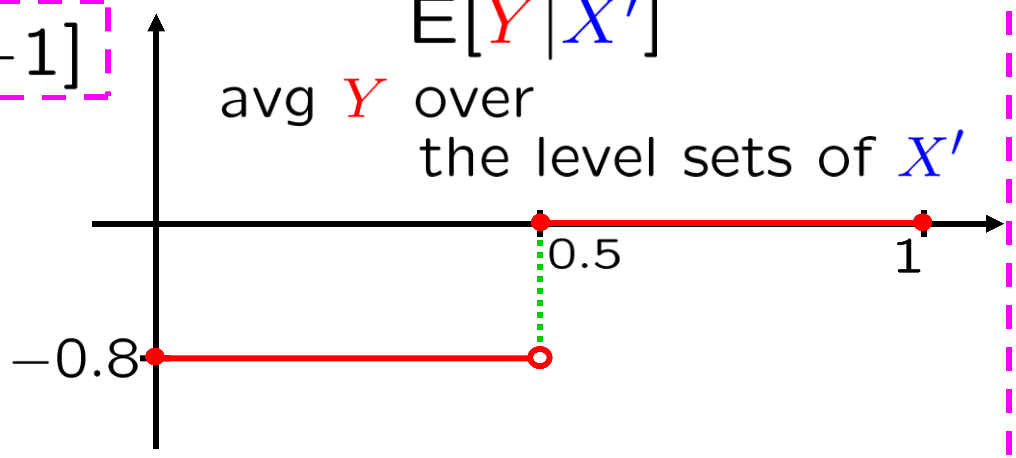
$E[Y|X' = -1]$

||

-0.8

$E[Y|X']$

avg Y over the level sets of X'



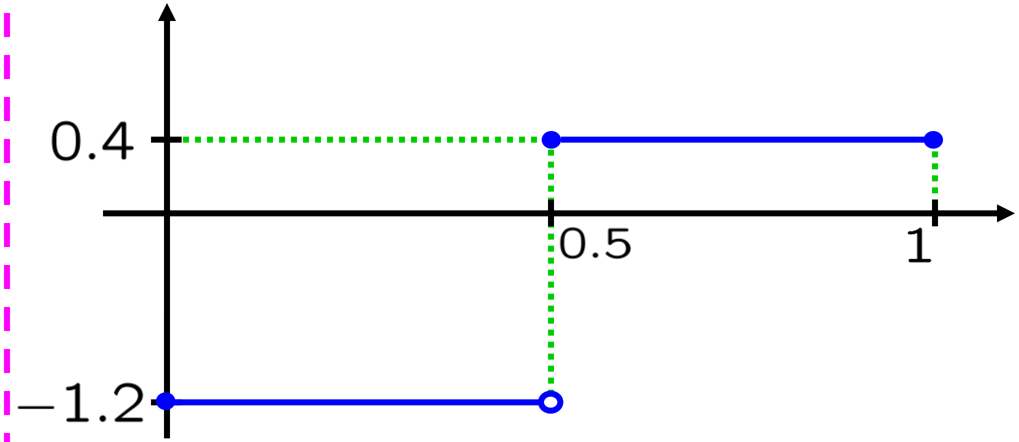
$E[Y|X' = 1]$

||

0

Conditional exp. of a PCRV, given a partition

X



$$X(\Omega) = \{-1.2, 0.4\}$$

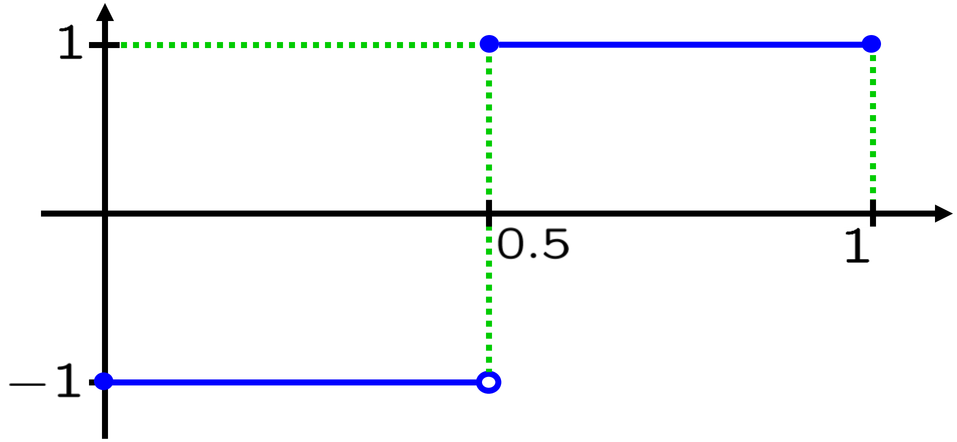
$$X^{-1}(-1.2) = [0, 0.5)$$
$$X^{-1}(0.4) = [0.5, 1]$$

Definition:
Let V be a PCRV. The **partition** **of** V is

$$\boxed{\mathcal{P}_V} := \{V^{-1}(y) \,|\, y \in V(\Omega)\}.$$

e.g.: $\mathcal{P} := \mathcal{P}_X = \{ \quad [0, 0.5) \quad , \quad [0.5, 1] \quad \}$
is the partition of X

X'



The partition of X or of X' is $\{[0, 0.5), [0.5, 1]\}$

$$X'(\Omega) = \{-1, 1\}$$

$$(X')^{-1}(-1) = [0, 0.5)$$
$$(X')^{-1}(1) = [0.5, 1]$$

Definition:
Let V be a PCRV. The **partition of** V is
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e.g.: $\mathcal{P} := \mathcal{P}_X = \{ \quad [0, 0.5) \quad , \quad [0.5, 1] \quad \}$
is the partition of X or X'.

$\mathcal{P}$

$\because$

$\|$

$\{[0, 0.5), [0.5, 1]\}$

The partition of X or of X' is

$\{[0, 0.5), [0.5, 1]\}$

$$X'(\Omega) = \{-1, 1\}$$

$$(X')^{-1}(-1) = [0, 0.5)$$

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Definition:

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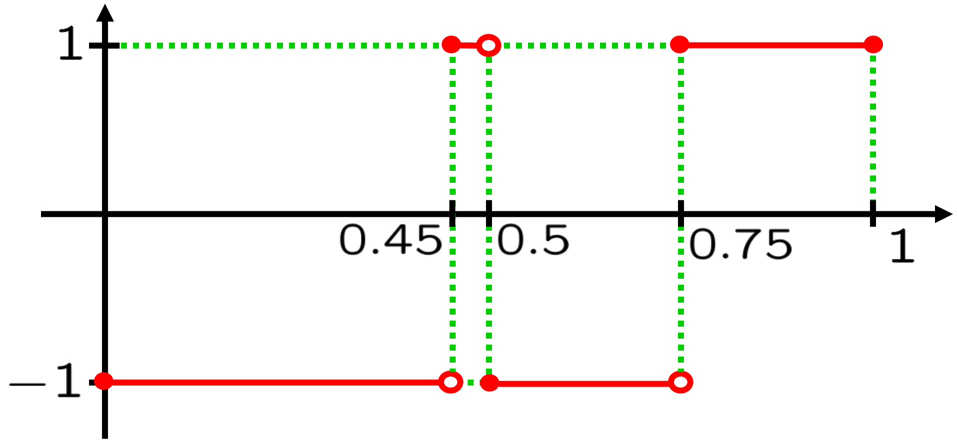
e.g.: $\mathcal{P} := \mathcal{P}_X = \{ \quad [0, 0.5) \quad , \quad [0.5, 1] \quad \}$ is the partition of X or X'.

$\mathcal{P}$ $\stackrel{\cdot\cdot}{=}$ $\{[0, 0.5), [0.5, 1]\}$

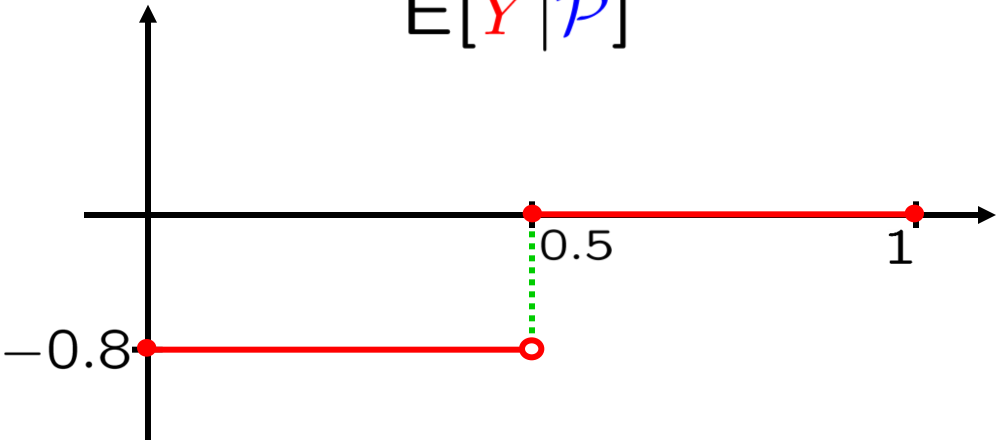
The partition of X or of X' is $\{[0, 0.5), [0.5, 1]\}$

Y



$E[Y|[0, 0.5)] = -0.8$

$E[Y|\mathcal{P}]$



$E[Y|[0.5, 1]] = 0$

Let V be a PCRV.

Definition: For any subinterval I of $[0,1]$,

$$|I| \neq 0 \Rightarrow \boxed{\mathrm{E}[V|I]} := \frac{1}{|I|}\int_I V(\omega)\, d\omega.$$

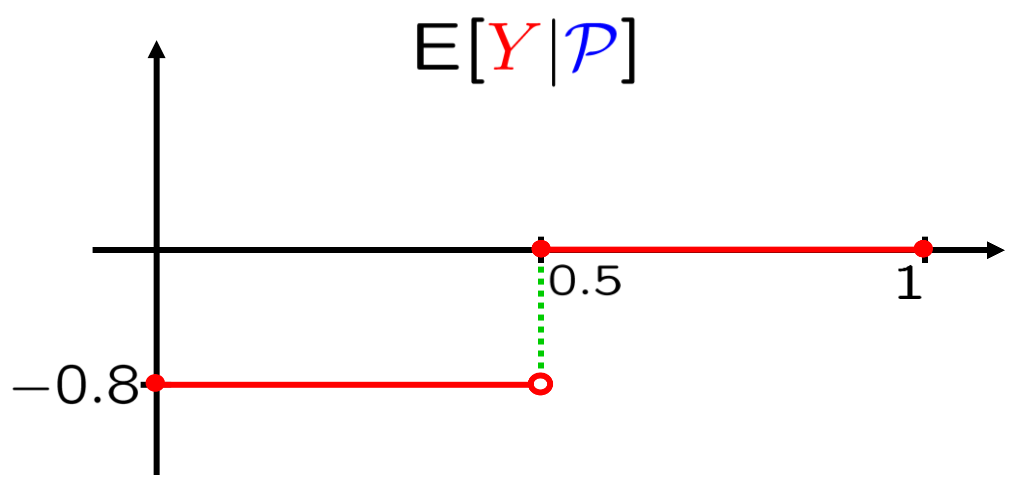
Definition:

Let $\mathcal{Q}$ be a partition of $\Omega := [0,1]$ by intervals.

$\forall \omega \in \Omega$, let $\boxed{Q_\omega}$ be the unique set in $\mathcal{Q}$ such that $\omega \in Q_\omega$.

Then $\boxed{\mathrm{E}[V|\mathcal{Q}]}$ is the fn $\omega \mapsto \mathrm{E}[V|Q_\omega] : \Omega \to \mathbb{R}$.

$$\mathcal{P} := \{ \quad [0, 0.5) \quad , \quad [0.5, 1] \quad \}$$



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Fact: $\mathsf{E}[V|\mathcal{Q}]$ is a PCRV.

Definition: Let V, W be PCRVs.

Let $\mathcal{P}$ be the partition of W.

Then $\mathsf{E}[V|W] := \mathsf{E}[V|\mathcal{P}]$.

Difficulty: Sets in $\mathcal{P}$ might not be intervals.

They might be fUofIs.

(finite unions of intervals)

Let V be a PCRV.

Definition: For any fUofI $I \subseteq [0,1]$,

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(finite unions of intervals)

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Definition:

Let $\mathcal{Q}$ be a partition of $\Omega := [0,1]$ by fUofIs.

of size > 0

$\forall \omega \in \Omega$, let Q_ω be the unique set in $\mathcal{Q}$ such that $\omega \in Q_\omega$.

Then $\mathsf{E}[V|\mathcal{Q}]$ is the fn $\omega \mapsto \mathsf{E}[V|Q_\omega] : \Omega \to \mathbb{R}$.

Fact: $\mathsf{E}[V|\mathcal{Q}]$ is a PCRV.

Definition: Let V, W be PCRVs.

Let $\mathcal{P}$ be the partition of W.

Assume all sets in $\mathcal{P}$ have size > 0.

Then $\mathsf{E}[V|W] := \mathsf{E}[V|\mathcal{P}]$.

New difficulty: Some sets in $\mathcal{P}$ might have size $= 0$.

More on this later.

a.e. constant := const. except at finitely many pts

Definition:

Let V be a PCRV.

Let $\mathcal{Q}$ be a partition of $[0,1]$.

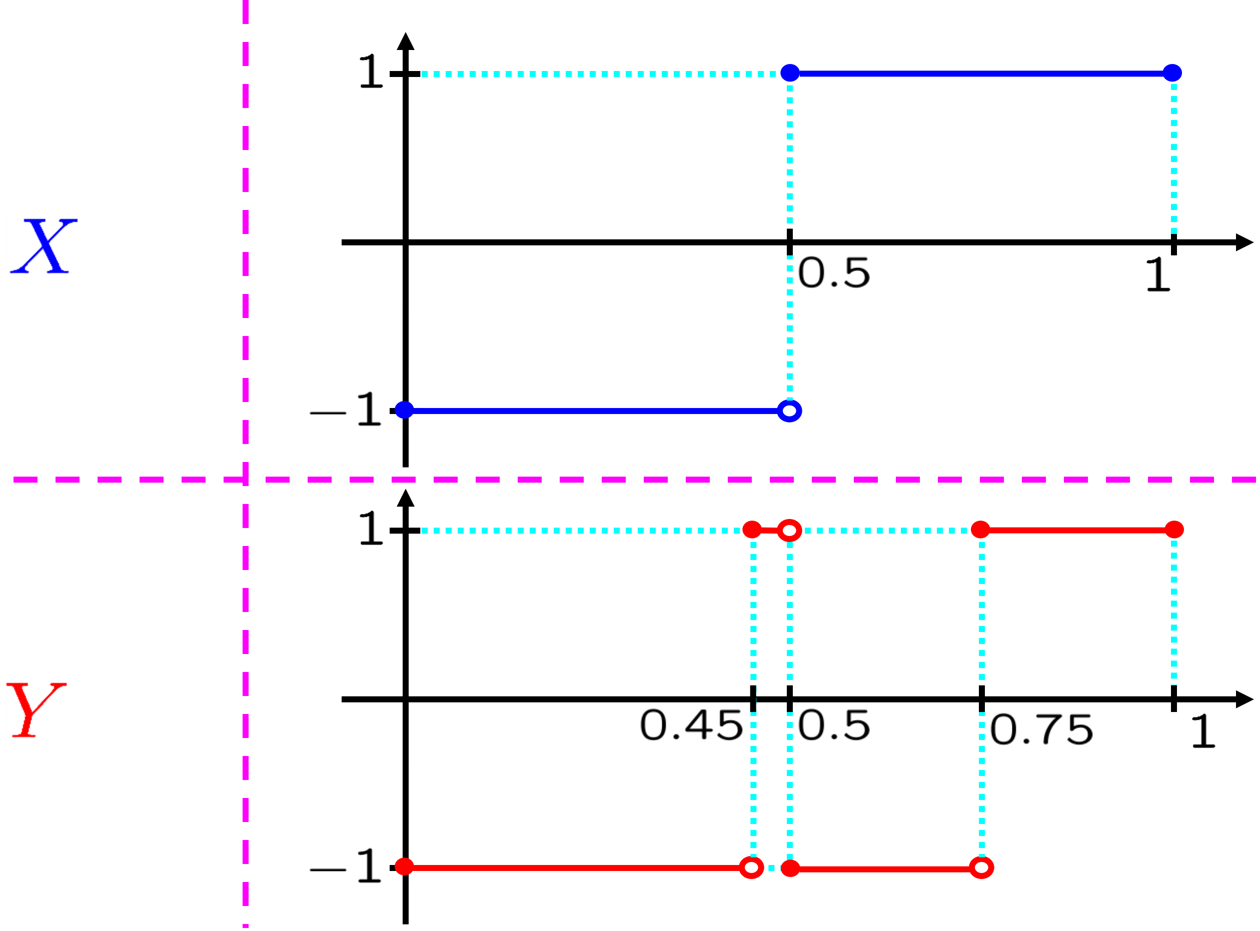
We say that V is $\mathcal{Q}$-**measurable**

if V is a.e. constant on each set in $\mathcal{Q}$.

Idea: If Tyche picks $\omega \in [0,1]$
and tells me which set in $\mathcal{Q}$ contains ω,
does that determine $V(\omega)$ a.s.?

Think of a partition as being like
a question we might pose to Tyche.
Its answer reveals information about ω,
but typically doesn't tell us ω exactly.

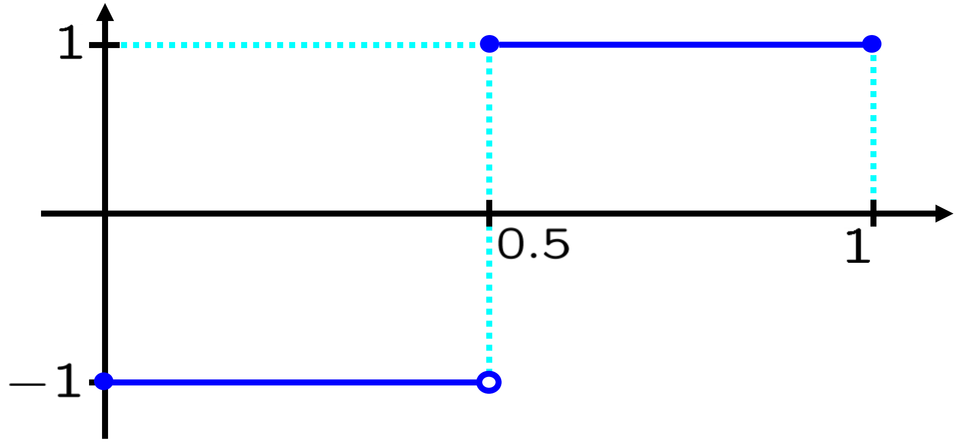
If the answer to "Which set in $\mathcal{Q}$ contains ω?"
a.s. gives enough info to answer "What is $V(\omega)$?",
then we say that V is $\mathcal{Q}$-measurable.



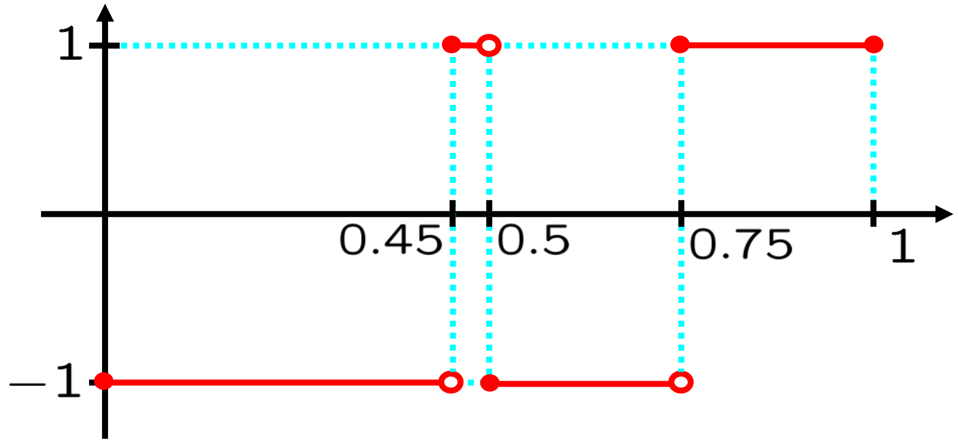
If not, we can try to find the "closest" PCRV to V that *is* $\mathcal{Q}$-measurable.

If the answer to "Which set in $\mathcal{Q}$ contains ω?" a.s. gives enough info to answer "What is $V(\omega)$?", then we say that V is $\mathcal{Q}$-measurable.

X



Y



$$\mathcal{P} := \mathcal{P}_X = \{ \quad [0, 0.5) \quad , \quad [0.5, 1] \quad \}$$

Then X is $\mathcal{P}$-measurable, but Y is not.

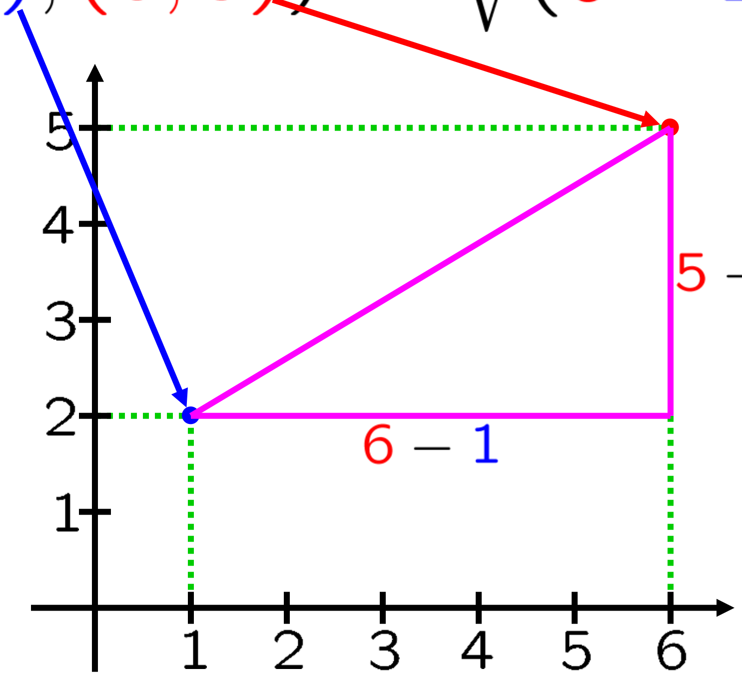
If we try to make Y into a $\mathcal{P}$-measurable PCRV, without changing its expectation on any set in $\mathcal{P}$, we get $\mathrm{E}[Y|\mathcal{P}]$.

Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.

In the plane,

$$\mathrm{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$



Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.

Let's start with an example.

In the plane,

$$\mathrm{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$

Def'n: For any two PCRVs, A and B,

$$\mathrm{dist}(A,B) := \sqrt{\int_0^1 ([A(\omega)] - [B(\omega)])^2 \quad d\omega}$$

$$= \sqrt{\mathrm{E}[\quad (A-B)^2 \quad]}$$

$$= \|A - B\|_{L^2}$$

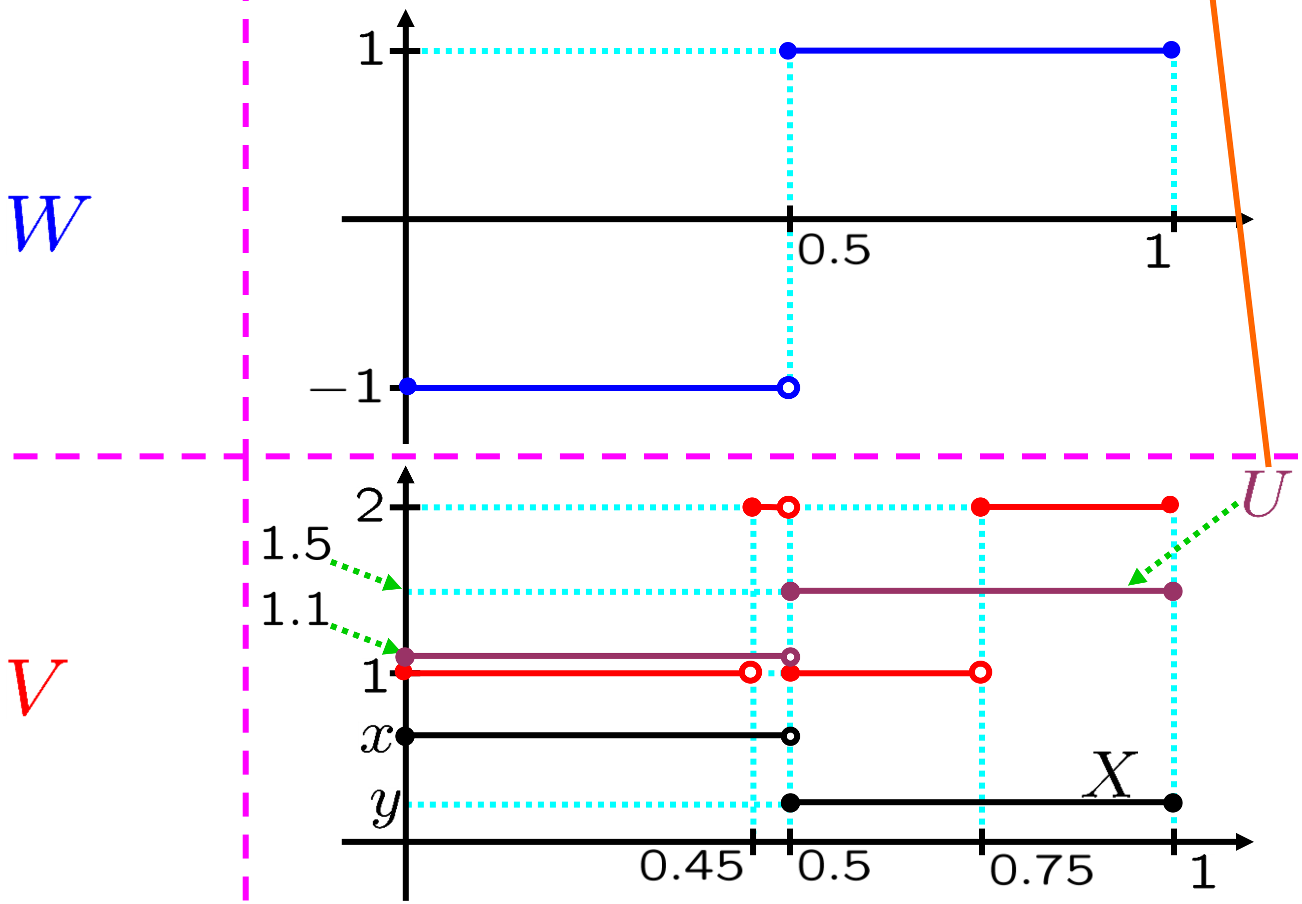
Def'n: For any PCRV X,

$$\|X\|_{L^2} := \sqrt{\mathrm{E}[\quad X^2 \quad]}$$

Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.

Let's start with an example.

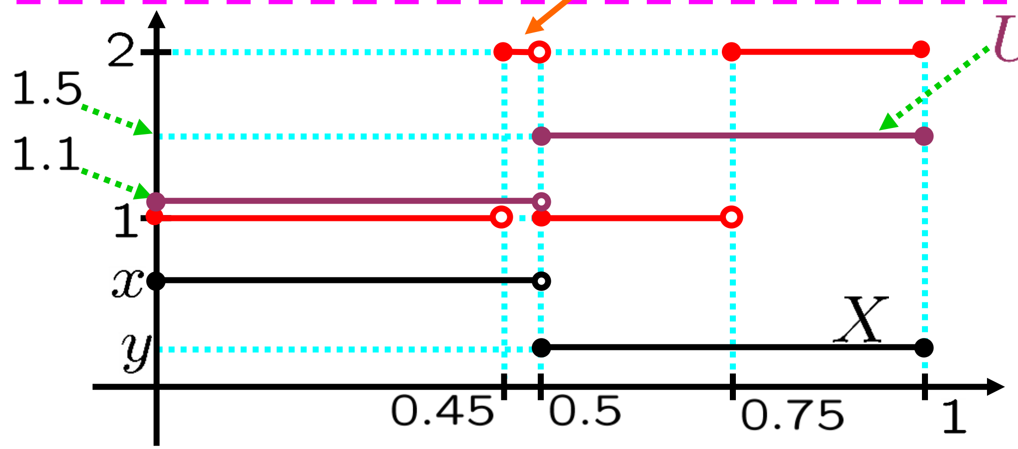


Goal:
$\forall \mathcal{P}_W$-msbl PCRVs X,
$\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

Average V over each of the level sets of W.

$$X = \begin{cases} x, & \text{on } [0, 0.45) \\ x, & \text{on } [0.45, 0.5) \\ y, & \text{on } [0.5, 0.75) \\ y, & \text{on } [0.75, 1] \end{cases} \qquad V = \begin{cases} 1, & \text{on } [0, 0.45) \\ 2, & \text{on } [0.45, 0.5) \\ 1, & \text{on } [0.5, 0.75) \\ 2, & \text{on } [0.75, 1] \end{cases}$$

V

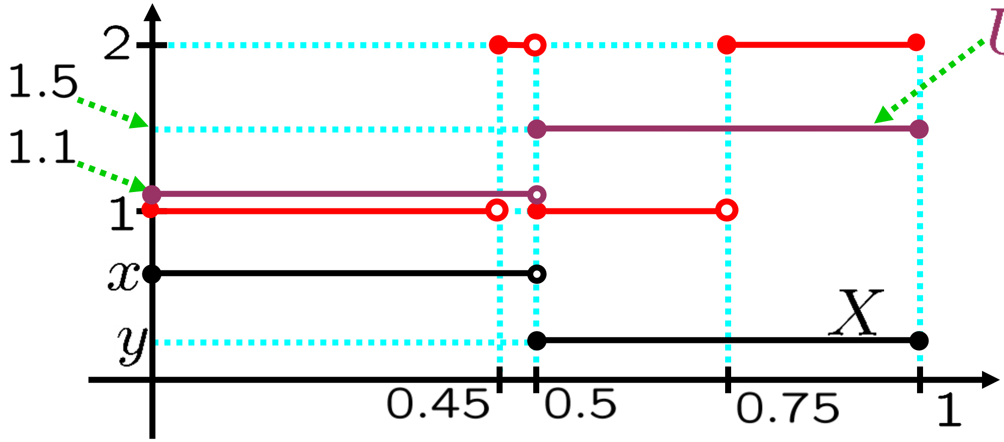


Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

$$X = \begin{cases} x, & \text{on } [0, 0.45) \\ x, & \text{on } [0.45, 0.5) \\ y, & \text{on } [0.5, 0.75) \\ y, & \text{on } [0.75, 1] \end{cases} \qquad V = \begin{cases} 1, & \text{on } [0, 0.45) \\ 2, & \text{on } [0.45, 0.5) \\ 1, & \text{on } [0.5, 0.75) \\ 2, & \text{on } [0.75, 1] \end{cases}$$

$$X - V = \begin{cases} x - 1, & \text{on } [0, 0.45) \\ x - 2, & \text{on } [0.45, 0.5) \\ y - 1, & \text{on } [0.5, 0.75) \\ y - 2, & \text{on } [0.75, 1] \end{cases}$$

V



Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

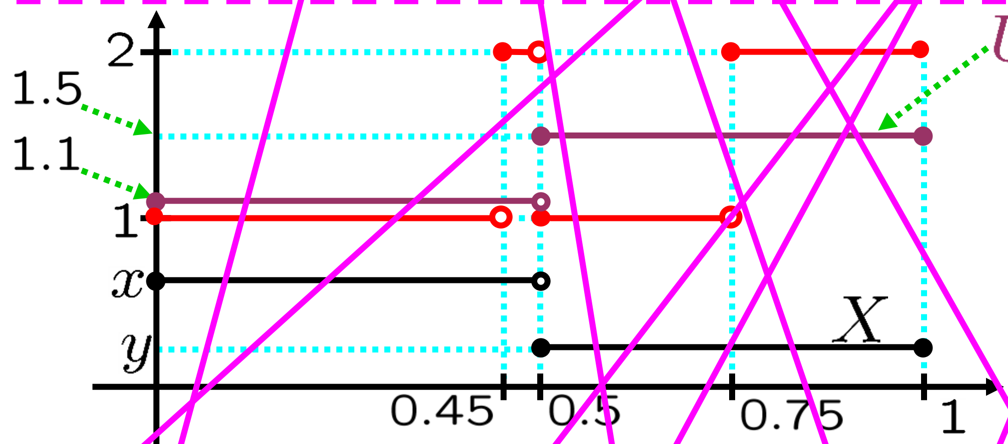
$\|X - V\|_{L^2}^2 = \mathrm{E}[(X - V)^2]$

$$(X - V)^2 = \begin{cases} (x-1)^2, & \text{on } [0, 0.45) \\ (x-2)^2, & \text{on } [0.45, 0.5) \\ (y-1)^2, & \text{on } [0.5, 0.75) \\ (y-2)^2, & \text{on } [0.75, 1] \end{cases}$$

0.45
0.05
0.25
0.25

$$X - V = \begin{cases} x-1, & \text{on } [0, 0.45) \\ x-2, & \text{on } [0.45, 0.5) \\ y-1, & \text{on } [0.5, 0.75) \\ y-2, & \text{on } [0.75, 1] \end{cases}$$

V



Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

$$\|X - V\|_{L^2}^2 = \mathrm{E}[(X - V)^2] = 0.45(x-1)^2 + 0.05(x-2)^2 + 0.25(y-1)^2 + 0.25(y-2)^2$$

$$0.45(x-1)^2+0.05(x-2)^2+0.25(y-1)^2+0.25(y-2)^2$$

$$0.9\,(x-1)\;+\;0.1\,(x-2) \;\|\; 0$$

$$(0.9+0.1)x=(0.9)1+(0.1)2$$

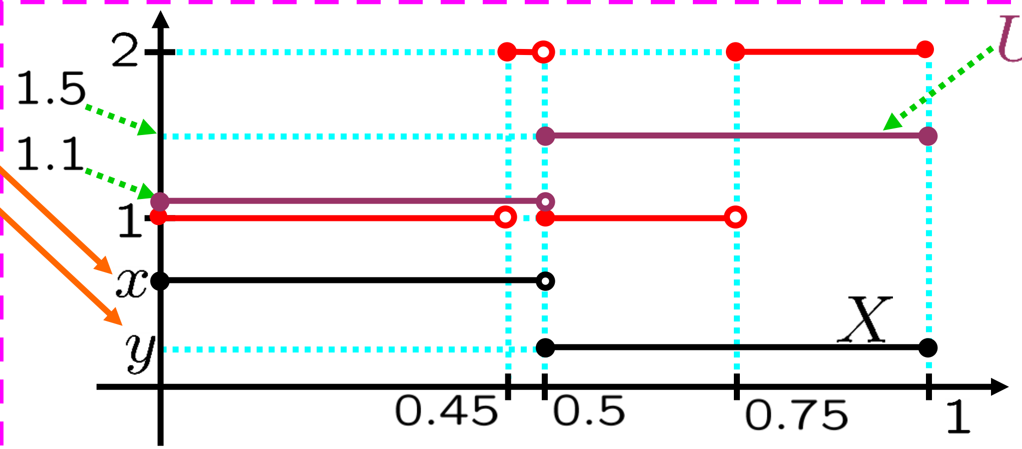
$$x=1.1$$

$$0.5\,(y-1)\;+\;0.5\,(y-2) \;\|\; 0$$

$$(0.5+0.5)y=(0.5)1+(0.5)2$$

$$y=1.5$$

V

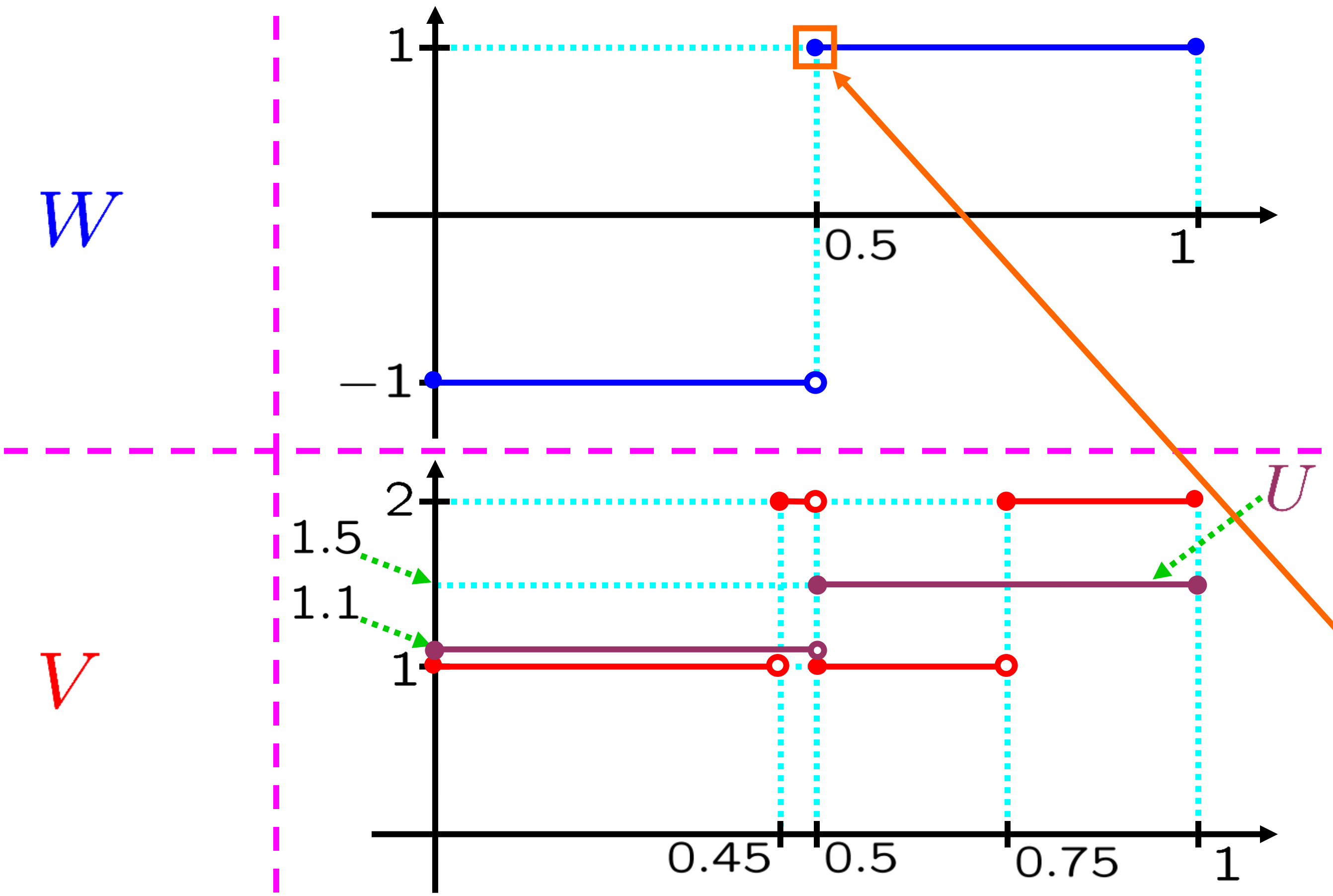


Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X-V\|_{L^2} \geq \|U-V\|_{L^2}$.

$$\|X-V\|_{L^2}^2=\mathrm{E}[(X-V)^2]=0.45(x-1)^2+0.05(x-2)^2+0.25(y-1)^2+0.25(y-2)^2$$

Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.



Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

Average V over each of the level sets of W.

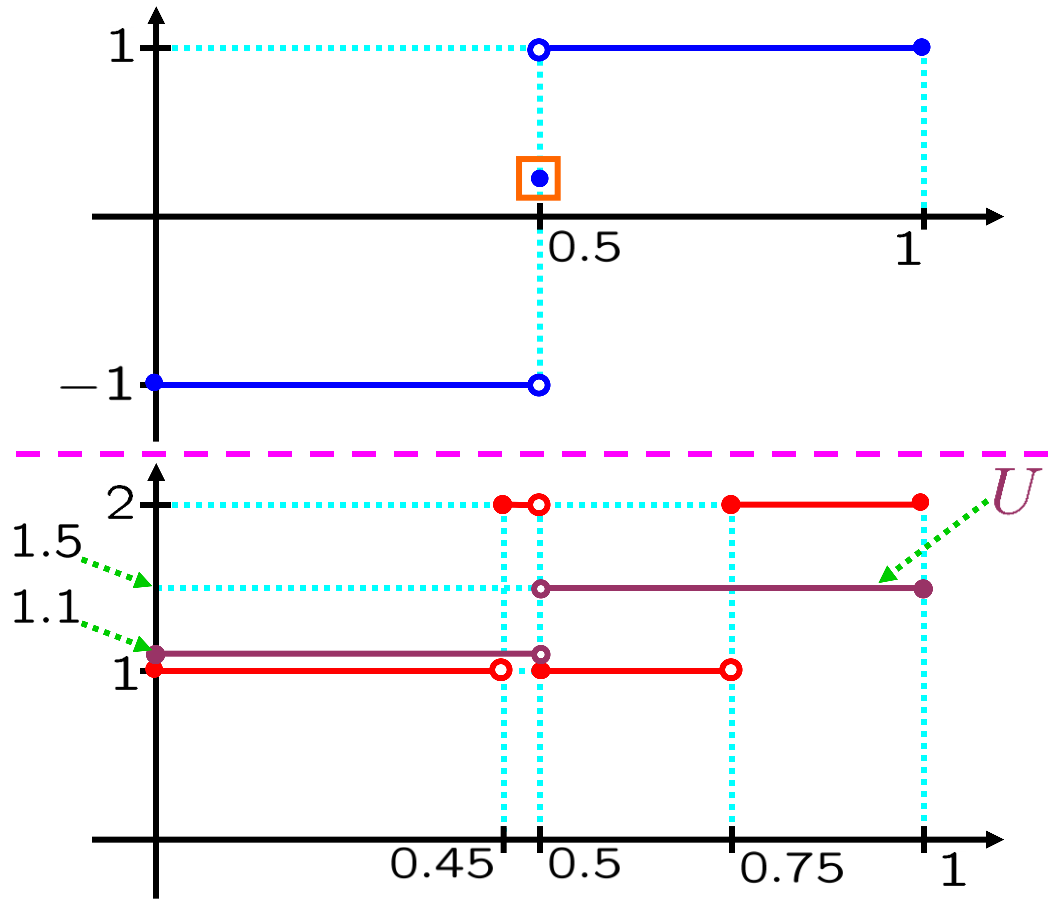
Problem: Some of the level sets may have length $= 0$.

Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.

W

V



Goal:
$\forall \mathcal{P}_W$-msbl PCRVs X,
$\|X - V\|_{L^2}$
$\geq$
$\|U - V\|_{L^2}$.

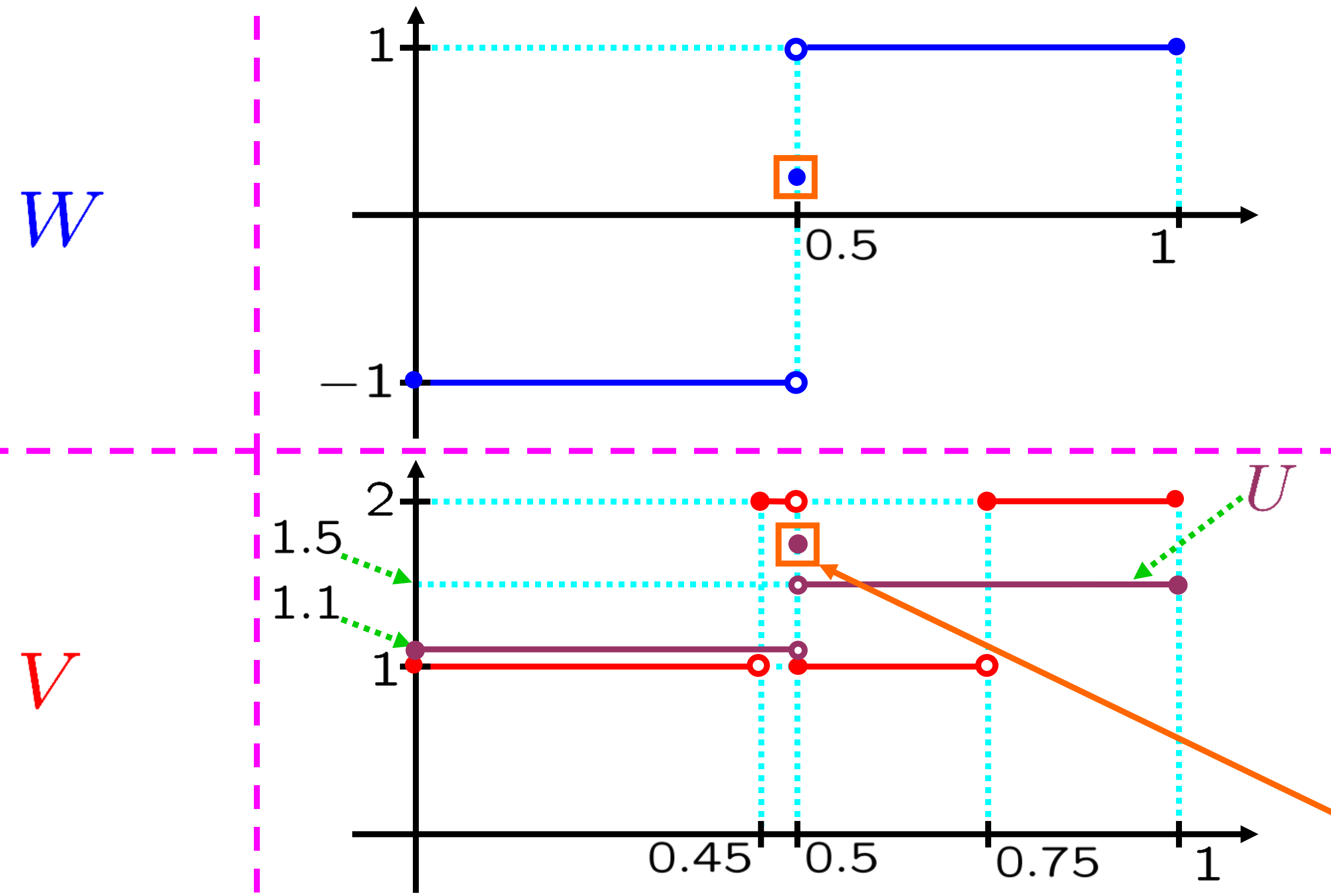
Average V over each of the level sets of W.

that have positive length.

Problem: Some of the level sets may have length $= 0$.

Let V be a PCRV. Let W be a PCRV.

Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.



Goal: $\forall \mathcal{P}_W$-msbl PCRVs X, $\|X - V\|_{L^2} \geq \|U - V\|_{L^2}$.

Average V over each of the level sets of W *that have positive length.*

Make any choice you want on the rest.

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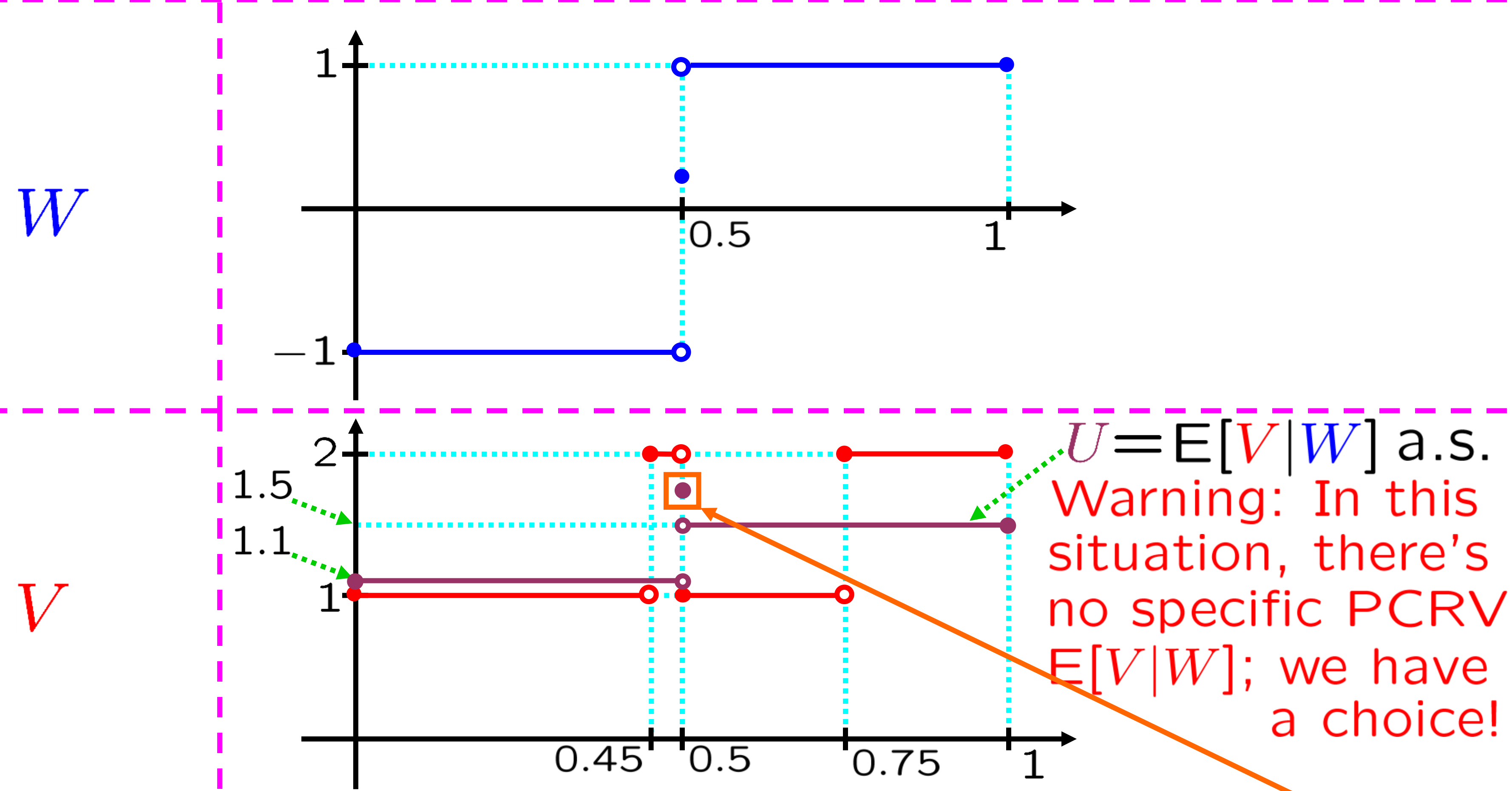

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Make any choice you want on the rest.

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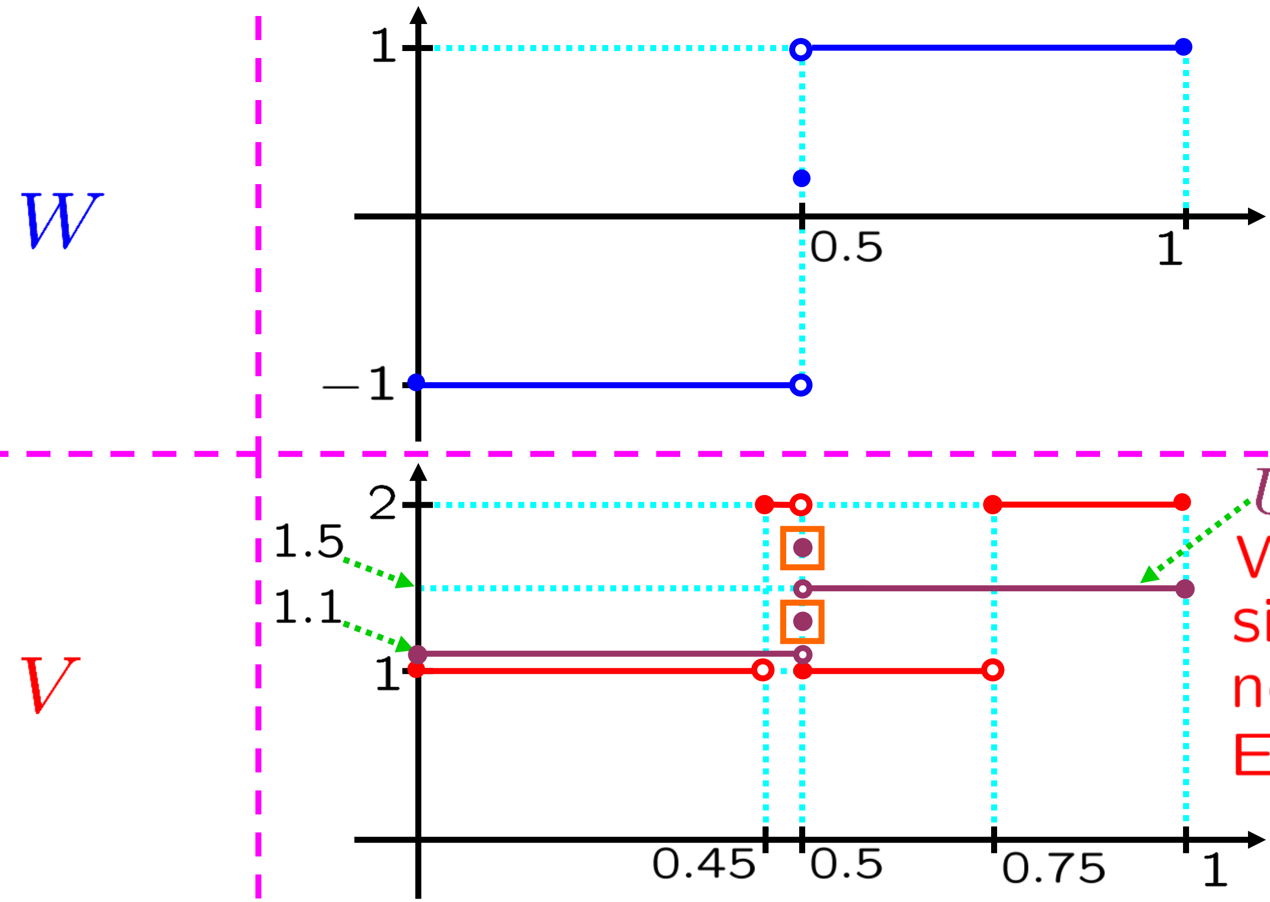
Goal: Interpret $\mathrm{E}[V|W]$ as a minimizer.



$U = \mathrm{E}[V|W]$ a.s.

Warning: In this situation, there's no specific PCRV $\mathrm{E}[V|W]$; we have a choice!

Average V over each of the level sets of W *that have positive length.*

Make any choice you want on the rest.

The Tower Law

Definition: Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[0,1]$.
We say that $\mathcal{P}$ is **finer than** $\mathcal{Q}$ if:
$\forall P \in \mathcal{P}$, $\exists Q \in \mathcal{Q}$ s.t. $P \subseteq Q$.

If "Which set in $\mathcal{P}$ contains ω?" gives enough info to answer "Which set in $\mathcal{Q}$ contains ω?", then we say that $\mathcal{P}$ is finer that $\mathcal{Q}$.

Note: $\mathcal{P}$ is finer than $\mathcal{Q}$ implies any $\mathcal{Q}$-measurable PCRV is $\mathcal{P}$-measurable.

- - -

If knowing $\mathcal{P}$ tells us $\mathcal{Q}$
and if knowing $\mathcal{Q}$ tells us the PCRV,
then knowing $\mathcal{P}$ tells us the PCRV.

The Tower Law

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The Tower Law: Let V be a PCRV.
Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[0,1]$ by fUofIs.
Assume that $\mathcal{P}$ is finer than $\mathcal{Q}$.
any $\mathcal{Q}$-measurable PCRV is $\mathcal{P}$-measurable.

The Tower Law

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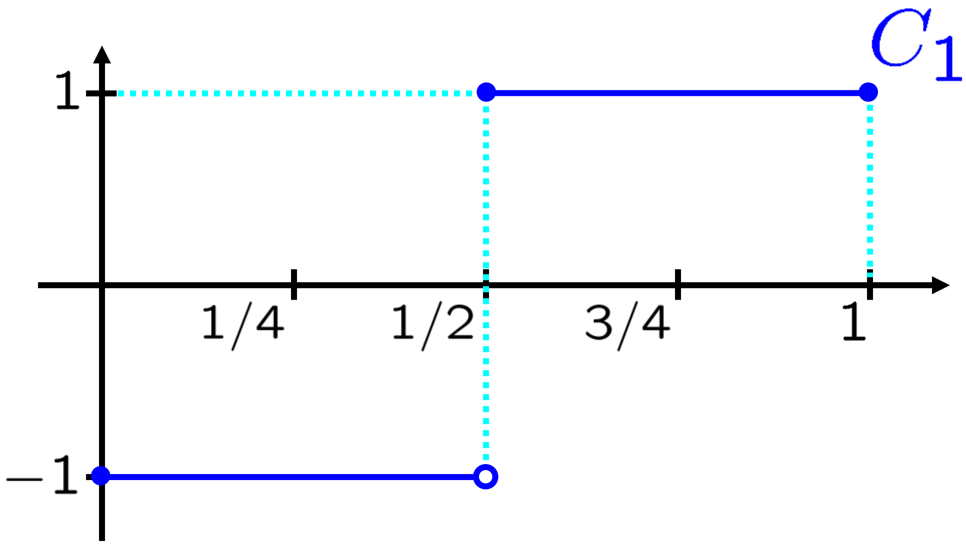
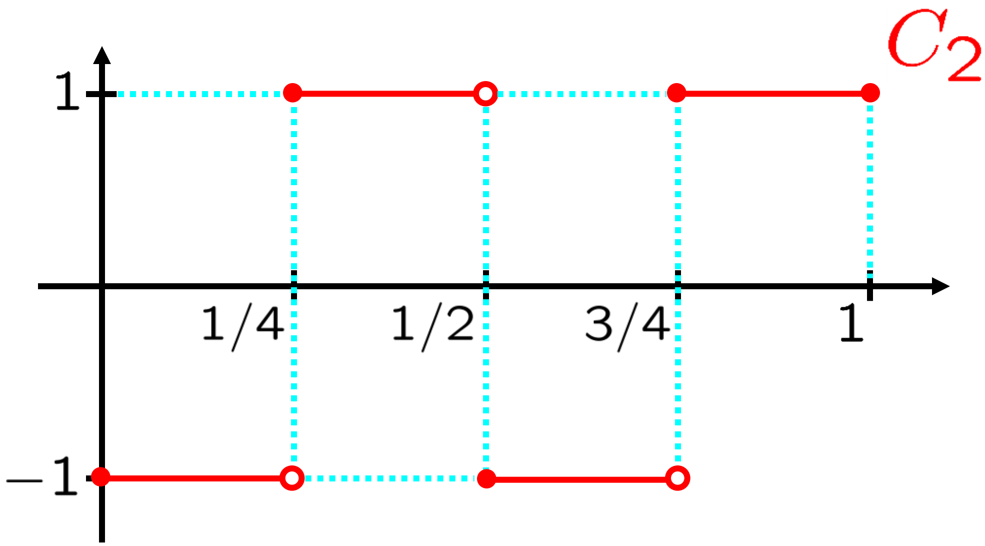
The Tower Law: Let V be a PCRV.
Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[0, 1]$ by fUofIs.
Assume that $\mathcal{P}$ is finer than $\mathcal{Q}$.
Then $\mathrm{E}[\ \mathrm{E}[V|\mathcal{P}]\ |\ \mathcal{Q}\ ] = \mathrm{E}[V|\mathcal{Q}]$.

The idea: Forcing $\mathcal{P}$-measurability is weaker than forcing $\mathcal{Q}$-measurability, so doing both is redundant.

PCRV





Distribution

C_1

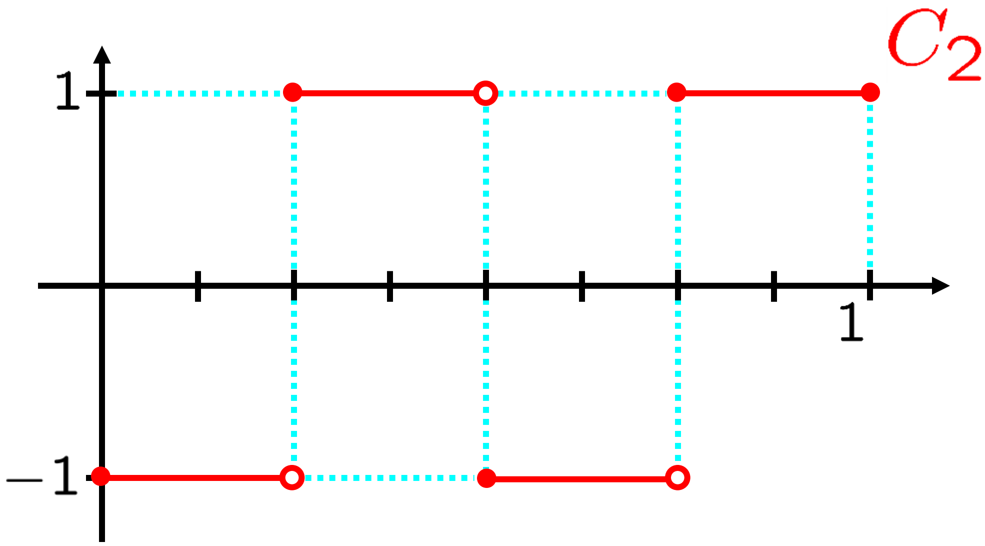
1	1/2
−1	1/2

C_2

1	1/2
−1	1/2

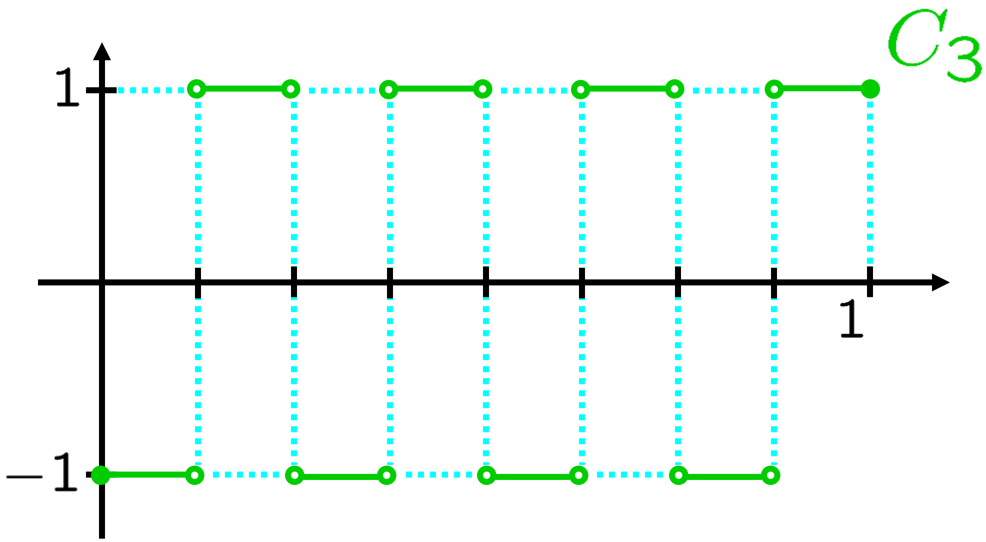
Note: C_1 and C_2 are identically distributed, but are not equal.

PCRV	Distribution



C_2

1	1/2
-1	1/2



C_3

1	1/2
-1	1/2

Note: C_1, C_2, C_3 are i.i.d. ← independent, identically distributed

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Continuing, we can form an i.i.d. sequence

$$C_1, C_2, C_3, C_4, C_5, \ldots$$

of PCRVs,

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Continuing, we can form an i.i.d. sequence

$$C_1, C_2, C_3, C_4, C_5, \ldots$$

of PCRVs, modeling a sequence of coin-flips:

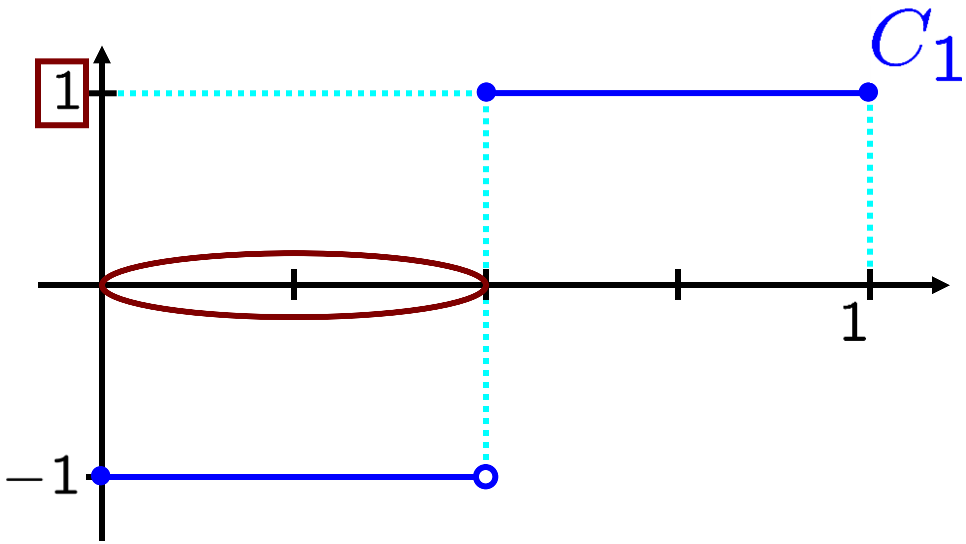
Tyche picks $\omega \in \Omega := [0, 1]$ at random.

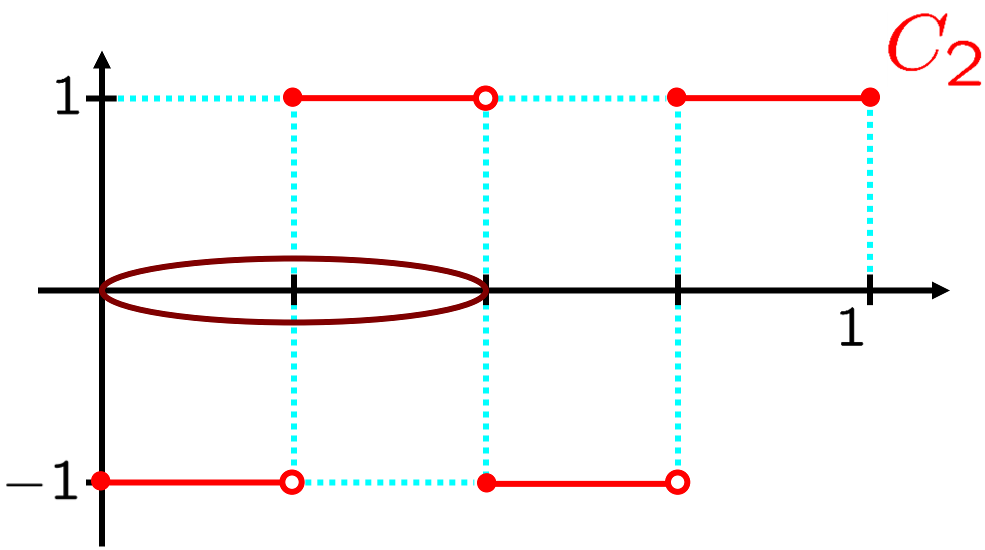
She reveals $C_1(\omega)$, and invites us to guess at $C_2(\omega)$.

Suppose she reveals that $C_1(\omega) = 1$.

Can you make an educated guess about $C_2(\omega)$?

PCRV





Distribution

C_1

1	1/2
−1	1/2

C_2

1	1/2
−1	1/2

Note: $\Pr[C_2 = -1 | C_1 = 1] = \Pr[C_2 = -1]$

Note: C_1, C_2, C_3 are i.i.d., but are distinct.

independent, identically distributed

Continuing, we can form an i.i.d. sequence

$$C_1, C_2, C_3, C_4, C_5, \ldots$$

of PCRVs, modeling a sequence of coin-flips:

Tyche picks $\omega \in \Omega := [0, 1]$ at random.

Knowledge of any finite number of $C_1(\omega), C_2(\omega), \ldots$ tells us nothing about any other.

Knowledge of any finite number of coin flips tells us nothing about any other.

e.g.: $X^{(1)} = C_1$,
$X^{(2)} = [C_1 + C_2]/\sqrt{2}$,
$X^{(3)} = [C_1 + C_2 + C_3]/\sqrt{3}$,
$\vdots$

Central Limit Theorem: test function

$\forall$continuous bounded $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{E}[f(X^{(n)})] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)][e^{-x^2/2}]\, dx.$$

Definition:

A **PCRV approximation** is a sequence $X^{(1)}, X^{(2)}, \ldots$ of PCRVs such that, $\forall$continuous, bounded $f : \mathbb{R} \to \mathbb{R}$, the sequence $\mathrm{E}[f(X^{(1)})], \mathrm{E}[f(X^{(2)})], \ldots$ is convergent.

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Key goal of probability theory:

Define **random variable** and **expectation** in such a way that, $\forall$PCRV approximation,

$$X^{(1)}, X^{(2)}, X^{(3)}, \ldots,$$

$\exists$a random variable Y such that $\forall$continuous, bounded $f : \mathbb{R} \to \mathbb{R}$,

$$\mathsf{E}[f(X^{(1)})], \mathsf{E}[f(X^{(2)})], \mathsf{E}[f(X^{(3)})], \ldots \to \mathsf{E}[f(Y)].$$

Definition:

We say $X^{(1)}, X^{(2)}, \ldots \to Y$ **in distribution**

if, $\forall$continuous, bounded $f : \mathbb{R} \to \mathbb{R}$,

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$\exists$a random variable Y such that

$$X^{(1)}, X^{(2)}, X^{(3)}, \ldots \to Y \text{ in distribution.}$$

e.g.: $X^{(1)} = C_1$,

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Remark: random variable

"Random variables" and "expectation" with sophisticated def'ns, give rise to a RV Z s.t., $\forall$continuous, exp-bdd $f : \mathbb{R} \to \mathbb{R}$,

standard normal RV

$$\mathrm{E}[f(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)][e^{-x^2/2}]\, dx.$$

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$$\mathsf{E}[f(X^{(1)})], \mathsf{E}[f(X^{(2)})], \mathsf{E}[f(X^{(3)})], \ldots \to \mathsf{E}[f(Z)]$$

$X^{(1)}, X^{(2)}, X^{(3)} \ldots \to Z$ in distribution.

$$\mathsf{E}[f(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)][e^{-x^2/2}]\, dx.$$

Deterministic conditional expectation

$$\mathsf{E}[X|Y] \text{ is deterministic}$$

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

$\Leftarrow$ is obvious

Pf of $\Rightarrow$:

$$\mathsf{E}[X|Y] = \mathsf{E}[\ \mathsf{E}[X|Y]\ ] \text{ a.s.}$$

$$= \mathsf{E}[X] \quad \text{QED}$$

Power Tower Law

Deterministic conditional expectation

X is independent of Y

implies $\quad \mathrm{E}[X|Y]$ is deterministic

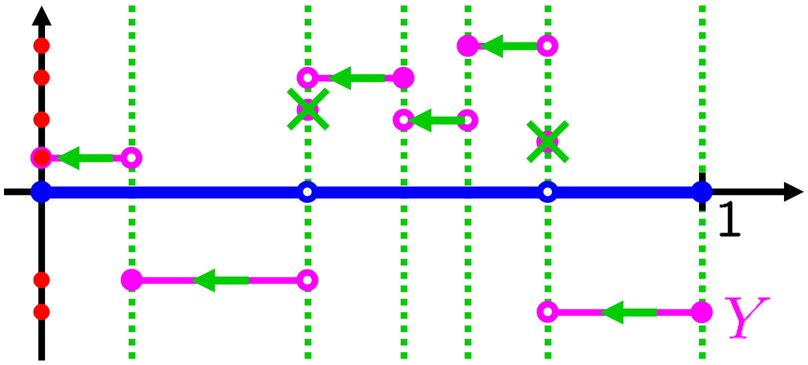
iff $\quad \mathrm{E}[X|Y] = \mathrm{E}[X]$ a.s.

Pf: Say X is independent of Y.

Want: $\mathrm{E}[X|Y] = \mathrm{E}[X]$ a.s.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$.

$$\Omega_0 := \bigcup_{b \in T} Y^{-1}(b)$$



Want: $\mathrm{E}[X|Y] = \mathrm{E}[X]$ on Ω_0

Deterministic conditional expectation

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Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on Ω_0

Fix $b \in T$.

Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on Ω_0

Deterministic conditional expectation

X is independent of Y

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Pf: Say X is independent of Y.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$.

$$\Omega_0 := \bigcup_{b \in T} Y^{-1}(b)$$

Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on Ω_0

Fix $b \in T$.

Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on $Y^{-1}(b)$

Deterministic conditional expectation

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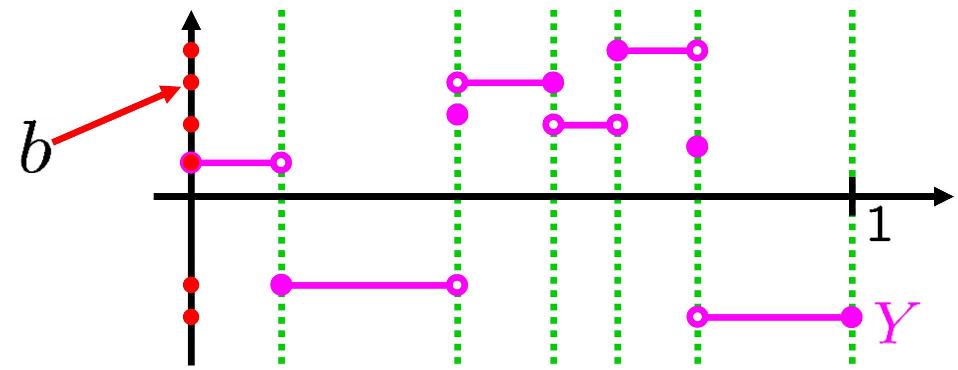


Fix $b \in T$.

Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on $Y^{-1}(b)$

Deterministic conditional expectation

X is independent of Y

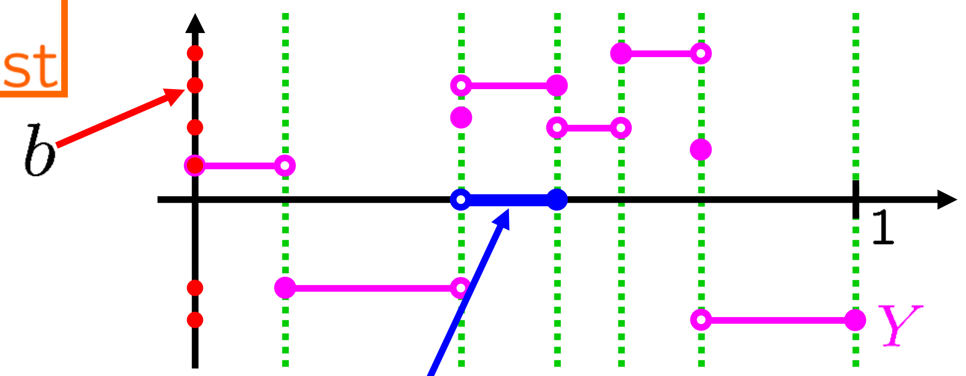
implies $\mathsf{E}[X|Y]$ is deterministic

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

Pf: Say X is independent of Y.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$. Fix $b \in T$.

For $\mathsf{E}[X|Y]$, avg X on the nonnull level sets of Y, and make choices on the rest



$$\frac{1}{|Y^{-1}(b)|} \int_{Y^{-1}(b)} X$$

Want: $\mathsf{E}[X|Y] = \mathsf{E}[X]$ on $Y^{-1}(b)$

Deterministic conditional expectation

X is independent of Y

implies $\mathsf{E}[X|Y]$ is deterministic

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

Pf: Say X is independent of Y. Let $S := X(\Omega)$.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$. Fix $b \in T$.

Want: $$\frac{1}{|Y^{-1}(b)|}\int_{Y^{-1}(b)} X = \mathsf{E}[X]$$

$$\frac{1}{|Y^{-1}(b)|}\int_{Y^{-1}(b)} X$$

$$Y^{-1}(b) = \coprod_{a \in S} [X^{-1}(a)] \cap [Y^{-1}(b)]$$

Wan

Deterministic conditional expectation

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Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$. Fix $b \in T$.

Want: $\frac{1}{|Y^{-1}(b)|} \int_{Y^{-1}(b)} X = \mathrm{E}[X]$

$X = a$ on $X^{-1}(a)$

$$\| $$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} \boxed{X}$$

$$\| $$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} \boxed{a}$$

Deterministic conditional expectation

X is independent of Y

implies $\mathsf{E}[X|Y]$ is deterministic

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

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Want: $\displaystyle\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} a = \mathsf{E}[X]$ $= \mathsf{E}[X]$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} a$$

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implies $\mathsf{E}[X|Y]$ is deterministic

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

Pf: Say X is independent of Y. Let $S := X(\Omega)$.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$. Fix $b \in T$.

Want: $$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} a = \mathsf{E}[X]$$

$$\| $$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} a \cdot \int_{[X^{-1}(a)] \cap [Y^{-1}(b)]} 1$$

$$\|$$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} a \cdot |[X^{-1}(a)] \cap [Y^{-1}(b)]|$$

Deterministic conditional expectation

X is independent of Y

implies $\mathsf{E}[X|Y]$ is deterministic

iff $\mathsf{E}[X|Y] = \mathsf{E}[X]$ a.s.

Pf: Say X is independent of Y. Let $S := X(\Omega)$.

Let $T := \{b \in \mathbb{R} \,|\, \Pr[Y = b] > 0\}$. Fix $b \in T$.

Want: $$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} a \cdot |[X^{-1}(a)] \cap [Y^{-1} = \mathsf{E}[X]\ \mathsf{E}[X]$$

$$\|$$

$$\frac{1}{\Pr[Y = b]} \sum_{a \in S} a \cdot \Pr[(X = a) \& (Y = b)]$$

$$\frac{1}{|Y^{-1}(b)|} \sum_{a \in S} a \cdot |[X^{-1}(a)] \cap [Y^{-1}(b)]|$$

Deterministic conditional expectation

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$$\| $$

$$\frac{1}{\Pr[Y = b]} \sum_{a \in S} a \cdot \Pr[(X = a) \& (Y = b)]$$

$$\| $$

$$\sum_{a \in S} a \cdot \Pr[(X = a) | (Y = b)]$$

Deterministic conditional expectation

X is independent of Y

implies $E[X|Y]$ is deterministic

iff $E[X|Y] = E[X]$ a.s.

Pf: Say X is independent of Y. Let $S := X(\Omega)$.

Let $T := \{b \in \mathbb{R} \mid \Pr[Y = b] > 0\}$. Fix $b \in T$.

Want: $\sum_{a \in S} a \cdot \Pr[(X = a)|(Y = b)] = E[X] = E[X]$

$E[X]$

$$\sum_{a \in S} a \cdot \Pr[(X = a)|(Y = b)]$$

Deterministic conditional expectation

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$X = a$ on $X^{-1}(a)$

$$\mathrm{E}[X] = \int_\Omega X = \sum_{a \in S} \int_{X^{-1}(a)} X = \sum_{a \in S} \int_{X^{-1}(a)} a$$

$$= \sum_{a \in S} a \cdot \int_{X^{-1}(a)} 1 = \sum_{a \in S} a \cdot |X^{-1}(a)|$$

Deterministic conditional expectation

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implies $\mathsf{E}[X|Y]$ is deterministic

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$\mathsf{E}[X]$

$$\mathsf{E}[X] = \sum_{a \in S} a \cdot |X^{-1}(a)| = \sum_{a \in S} a \cdot \Pr[X = a]$$

$$= \sum_{a \in S} a \cdot |X^{-1}(a)|$$

Deterministic conditional expectation

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$\mathrm{E}[X]$

Know: $\displaystyle\sum_{a \in S} a \cdot \mathrm{P} = \sum_{a \in S} a \cdot \Pr[X = a]$

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Want: $\sum_{a \in S} a \cdot \Pr[(X = a)|(Y = b)] = \mathsf{E}[X]$

equal QED

Know: $\sum_{a \in S} a \cdot \Pr[X = a] = \mathsf{E}[X]$

Deterministic conditional expectation

	X is independent of Y
implies	$\mathrm{E}[X\|Y]$ is deterministic
iff	$\mathrm{E}[X\|Y] = \mathrm{E}[X]$ a.s.
IOU implies	$\mathrm{E}[XY] = (\mathrm{E}[X])(\mathrm{E}[Y])$
iff	$\mathrm{Cov}[X, Y] = 0$
iff	X and Y are uncorrelated

$$\mathrm{Cov}[X, Y] = (\mathrm{E}[XY]) - (\mathrm{E}[X])(\mathrm{E}[Y])$$

Deterministic conditional expectation is logically "between" independent and uncorrelated.

Deterministic conditional expectation

X is independent of Y

implies E$[X|Y]$ is deterministic

E$[X|Y] =$ E$[X]$ a.s.

IOU implies E$[XY] = ($E$[X])($E$[Y])$

Does E$[X|Y]$ constant

imply X is independent of Y?

NO.

Does E$[XY] = ($E$[X])($E$[Y])$

imply E$[X|Y] =$ E$[X]$?

NO.

Deterministic conditional expectation

$$\mathsf{E}[X|Y] = \mathsf{E}[X] \text{ a.s.}$$

IOU implies $\mathsf{E}[XY] = (\mathsf{E}[X])(\mathsf{E}[Y])$

Pf:
$$\begin{aligned}\mathsf{E}[XY] &= \mathsf{E}[\ \mathsf{E}[(XY)|Y]\ ] \\ &= \mathsf{E}[\ Y \cdot \mathsf{E}[X|Y]\ ] \\ &= \mathsf{E}[\ Y \cdot \mathsf{E}[X]\ ] \\ &= \mathsf{E}[X] \cdot \mathsf{E}[Y]\end{aligned}$$
QED

Fact: If A is measurable w.r.t. the partition of C,
IOU then $\mathsf{E}[(AB)|C] = A \cdot \mathsf{E}[B|C]$ a.s.

Taking out what you know

E[•] is $\mathbb{R}$-linear,

i.e., $\mathsf{E}[sX + tY] = s(\mathsf{E}[X]) + t(\mathsf{E}[Y])$,

$\forall s, t \in \mathbb{R}$, $\forall$PCRVs X, Y.

Let $\mathcal{P}$ be a finite part'n of Ω by pos msr sets.

Let $\mathcal{F} := \{\mathcal{P}$-measurable PCRVs$\}$.

Fact: $\mathsf{E}[\bullet|\mathcal{P}]$ is $\mathcal{F}$-linear,

IOU

i.e., $\mathsf{E}[(FX+GY)|\mathcal{P}] = F(\mathsf{E}[X|\mathcal{P}])+G(\mathsf{E}[Y|\mathcal{P}])$,

$\forall F, G \in \mathcal{F}$, $\forall$PCRVs X, Y.

e.g.: $\mathcal{P} = \{\Omega\}$

e.g.: $G = Y = 0$

$\forall$RV X, $\forall \omega \in \Omega$, $(\mathsf{E}[X|\mathcal{P}])(\omega) = \mathsf{E}[X]$

$\mathcal{F} = \{$constant functions on $\Omega\}$

Fact: If A is measurable w.r.t. the partition of C,

IOU

then $\mathsf{E}[(AB)|C] = A \cdot \mathsf{E}[B|C]$ a.s.

Taking out what you know

$\mathrm{E}[(F X)|\mathcal{P}] = F(\mathrm{E}[X|\mathcal{P}])$, if F is $\mathcal{P}$-measurable.

$F :\to A,\ X :\to B,\ \mathcal{P} :\to \mathcal{P}_C$

If you know $\mathcal{P}$, then you know F, and you can "take out what you know".

Let $\mathcal{P}$ be a finite part'n of Ω by pos msr sets.
Let $\mathcal{F} := \{\mathcal{P}\text{-measurable PCRVs}\}$.

Fact: $\mathrm{E}[\bullet|\mathcal{P}]$ is $\mathcal{F}$-linear,
IOU

i.e., $\mathrm{E}[(FX+GY)|\mathcal{P}] = F(\mathrm{E}[X|\mathcal{P}])+G(\mathrm{E}[Y|\mathcal{P}])$,
$\forall F, G \in \mathcal{F}$, $\forall$PCRVs X, Y.

e.g.: $\mathcal{P} = \{\Omega\}$ | *e.g.*: $G = Y = 0$

$\forall$RV X, $\forall \omega \in \Omega$, $(\mathrm{E}[X|\mathcal{P}])(\omega) = \mathrm{E}[X]$

$\mathcal{F} = \{\text{constant functions on } \Omega\}$

Fact: If A is measurable w.r.t. the partition of C,
IOU
then $\mathrm{E}[(AB)|C] = A \cdot \mathrm{E}[B|C]$ a.s.

Taking out what you know

$E[(FX)|\mathcal{P}] = F(E[X|\mathcal{P}])$, if F is $\mathcal{P}$-measurable.

If you know $\mathcal{P}$, then you know F,
and you can "take out what you know".

Let $\mathcal{P}$ be a finite part'n of Ω by pos msr sets.
Let $\mathcal{F} := \{\mathcal{P}$-measurable PCRVs$\}$.

Fact: $E[\bullet|\mathcal{P}]$ is $\mathcal{F}$-linear,

IOU

I'll do this. $E[FX|\mathcal{P}]$ + You do this. $E[GY|\mathcal{P}]$ Want to take out both

i.e., $E[(FX+GY)|\mathcal{P}] = F(E[X|\mathcal{P}])+G(E[Y|\mathcal{P}])$,
$\forall F,G \in \mathcal{F}$, $\forall$PCRVs X,Y.

Pf: Let $F \in \mathcal{F}$, so F is const on sets in $\mathcal{P}$.

Want: $E[(FX)|\mathcal{P}] = F(E[X|\mathcal{P}])$

Fix $\omega_0 \in \Omega$. Choose $P_0 \in \mathcal{P}$ s.t. $\omega_0 \in P_0$

Want: $(E[(FX)|\mathcal{P}])(\omega_0) = [F(\omega_0)][(E[X|\mathcal{P}])(\omega_0)]$

Want: $E[(FX)|P_0] = [F(\omega_0)][\ E[X|P_0]\ ]$

Taking out what you know

$s_0 := F(\omega_0)$ $\qquad\qquad \forall \omega \in P_0, \quad F(\omega) = s_0$

$$\mathsf{E}[(FX)|P_0] = \mathsf{E}[(s_0 X)|P_0]$$
$$= s_0[\mathsf{E}[X|P_0]]$$
$$= [F(\omega_0)][\mathsf{E}[X|P_0]] \text{ QED}$$

STOP

Let $\mathcal{P}$ be a finite part'n of Ω by pos msr sets.
Let $\mathcal{F} := \{$RVs constant on sets in $\mathcal{P}\}$.

Fact: $\mathsf{E}[\bullet|\mathcal{P}]$ is $\mathcal{F}$-linear,

i.e., $\mathsf{E}[(FX+GY)|\mathcal{P}] = F(\mathsf{E}[X|\mathcal{P}])+G(\mathsf{E}[Y|\mathcal{P}])$,
$\forall F, G \in \mathcal{F}$, $\forall$PCRVs X, Y.

Pf: Let $F \in \mathcal{F}$, so F is const on sets in $\mathcal{P}$.

Want: $\mathsf{E}[(FX)|\mathcal{P}] = F(\mathsf{E}[X|\mathcal{P}])$ on this, F is const.

Fix $\omega_0 \in \Omega$. Choose $P_0 \in \mathcal{P}$ s.t. $\omega_0 \in P_0$

Want: $(\mathsf{E}[(FX)|\mathcal{P}])(\omega_0) = [F(\omega_0)][(\mathsf{E}[X|\mathcal{P}])(\omega_0)]$

Want: $\mathsf{E}[(FX)|P_0] = [F(\omega_0)][\ \mathsf{E}[X|P_0]\ ]$