Financial Mathematics

Stirling’s Formula

Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}\right)^n \to e^5$, as $n\to\infty$

Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}\right)^n \to e^5$

Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3}\right)^n \to e^5$

Fact: $x_n \sim \frac{5}{n} \quad \Rightarrow \quad (1+x_n)^n \to e^5$

Goal: Asymptotics of $n!$

Def'n: Say, $\forall n,\ a_n, b_n > 0$. Then $\boxed{a_n \sim b_n}$ means: $a_n/b_n \to 1$.

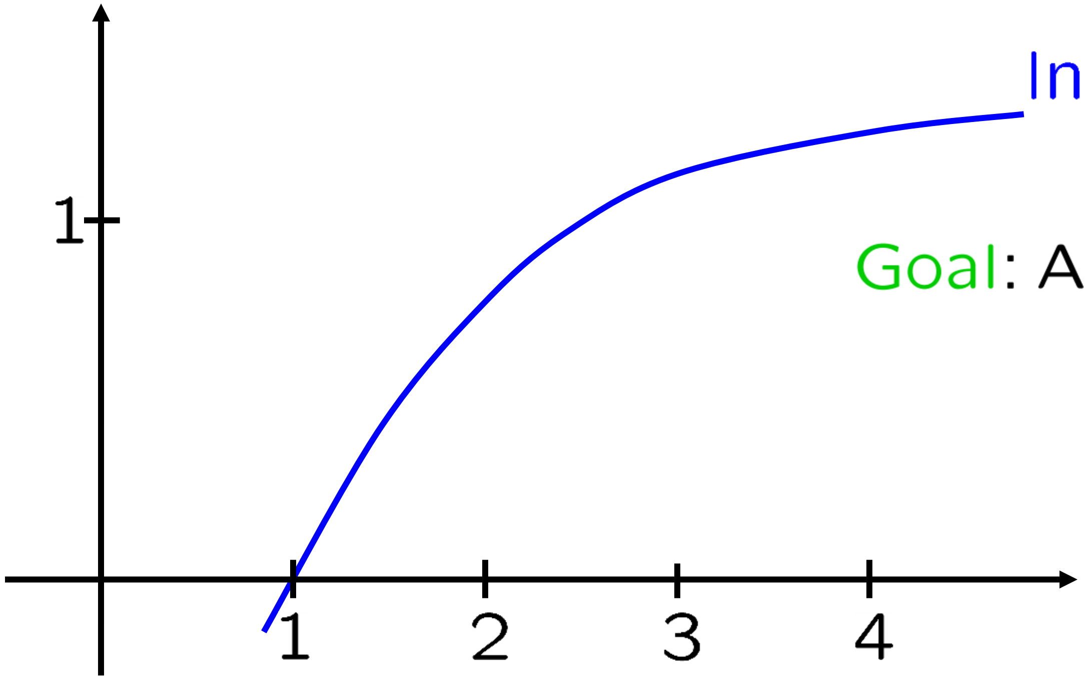
$$\tfrac{1}{2}n^3 + 500n^2 \sim \tfrac{1}{2}n^3$$

$$\frac{500n^2}{\frac{1}{2}n^3} \to 0$$

$$\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3} \sim \frac{5}{n}$$

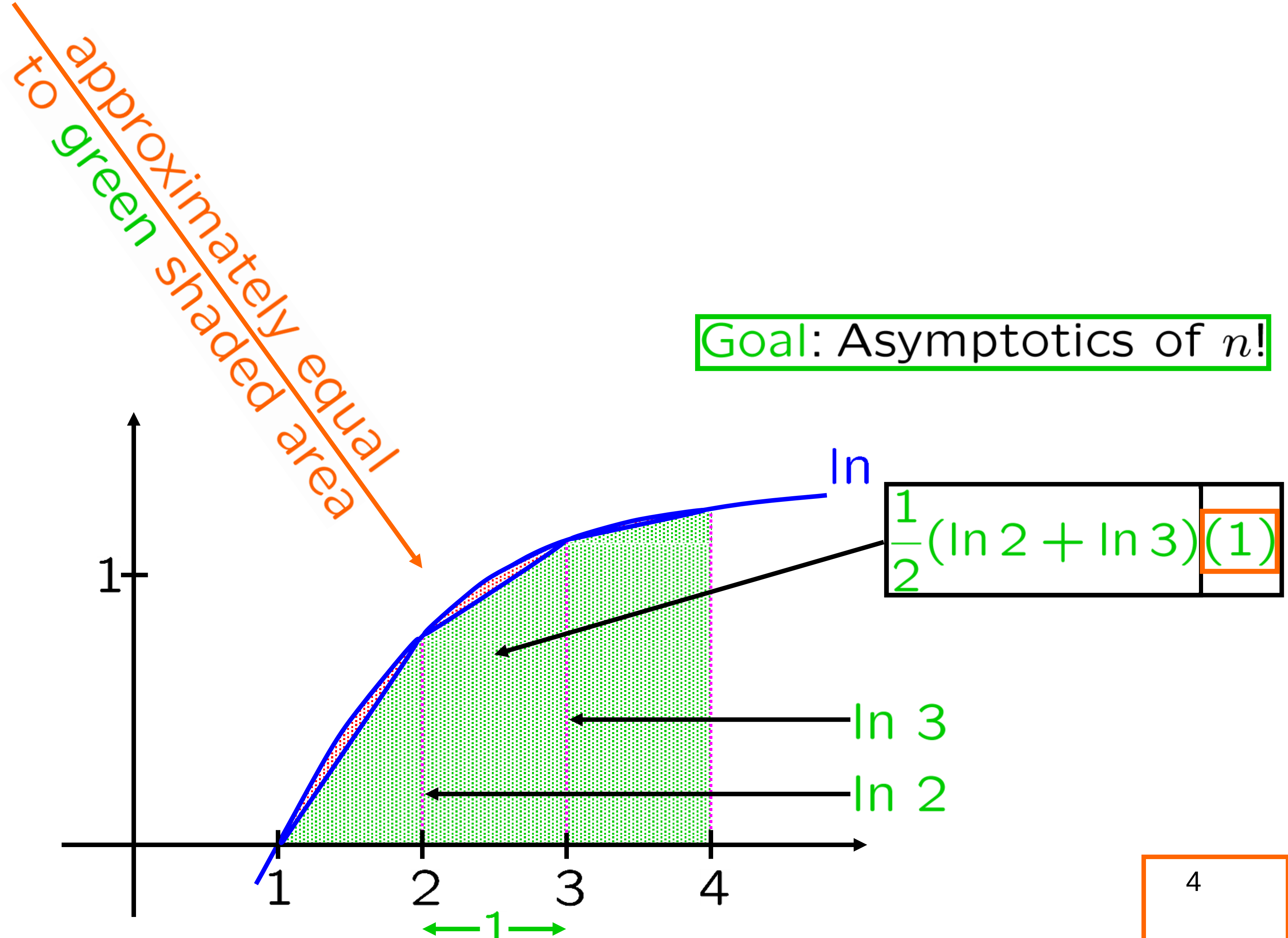
$$\frac{5}{n}+\frac{10{,}000}{n^2} \sim \frac{5}{n}$$

$\boxed{\text{Goal: Asymptotics of } n!}$



Goal: Asymptotics of $n!$

$$\int_1^4 \ln x \, dx$$



approximately equal
to green shaded area
Goal: Asymptotics of $n!$
ln
$\frac{1}{2}(\ln 2 + \ln 3)$ (1)
ln 3
ln 2
1
1
2
3
4
1

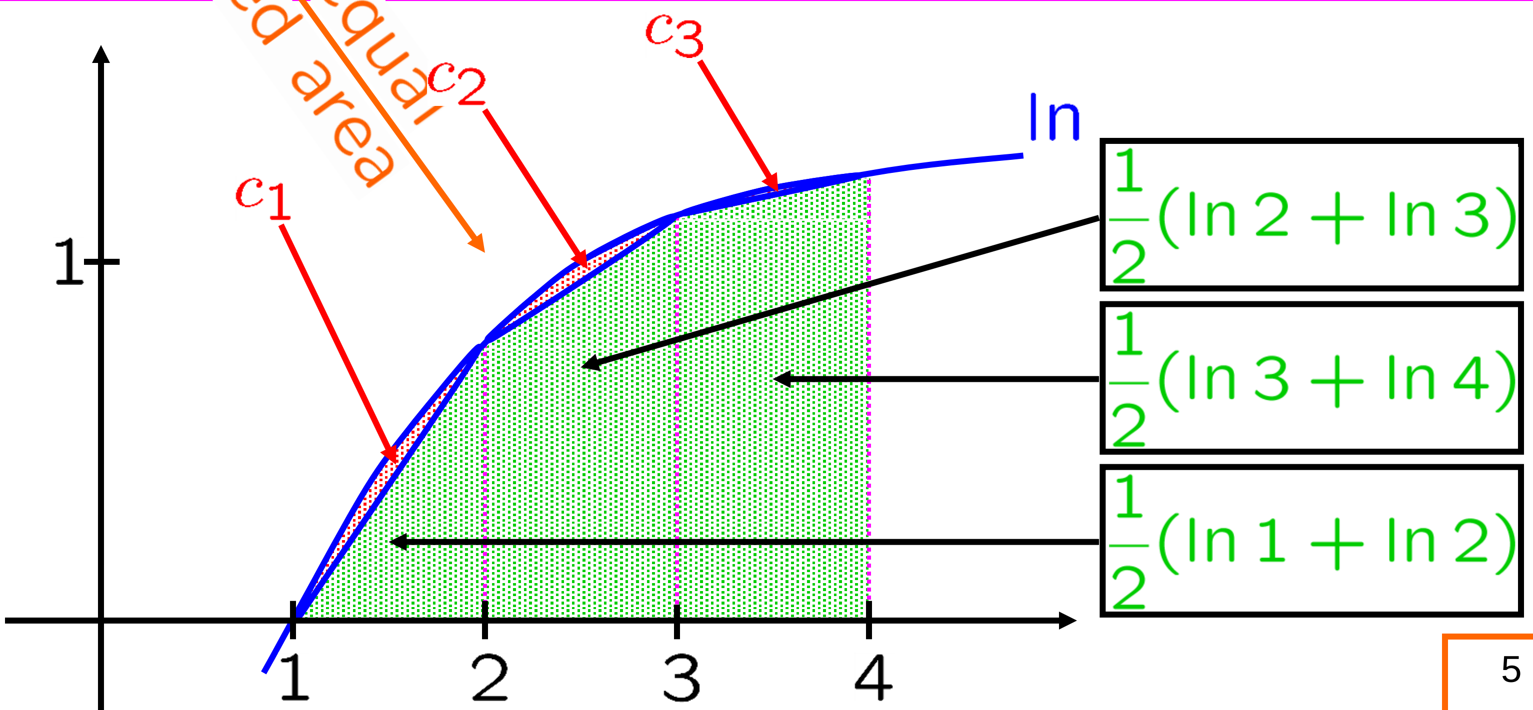
$$\int_1^4 \ln x\,dx \approx \boxed{\frac{1}{2}}(\boxed{\ln 1}+\boxed{\ln 2}) + \boxed{\frac{1}{2}}(\boxed{\ln 2}+\boxed{\ln 3}) + \boxed{\frac{1}{2}}(\boxed{\ln 3}+\boxed{\ln 4})$$
$$+c_1+c_2+c_3$$

zero

approximately equal to green shaded area

$$= \boxed{\ln 1}+\boxed{\ln 2}+\boxed{\ln 3}+\boxed{\ln 4-\frac{\ln 4}{2}}$$
$$+c_1+c_2+c_3$$

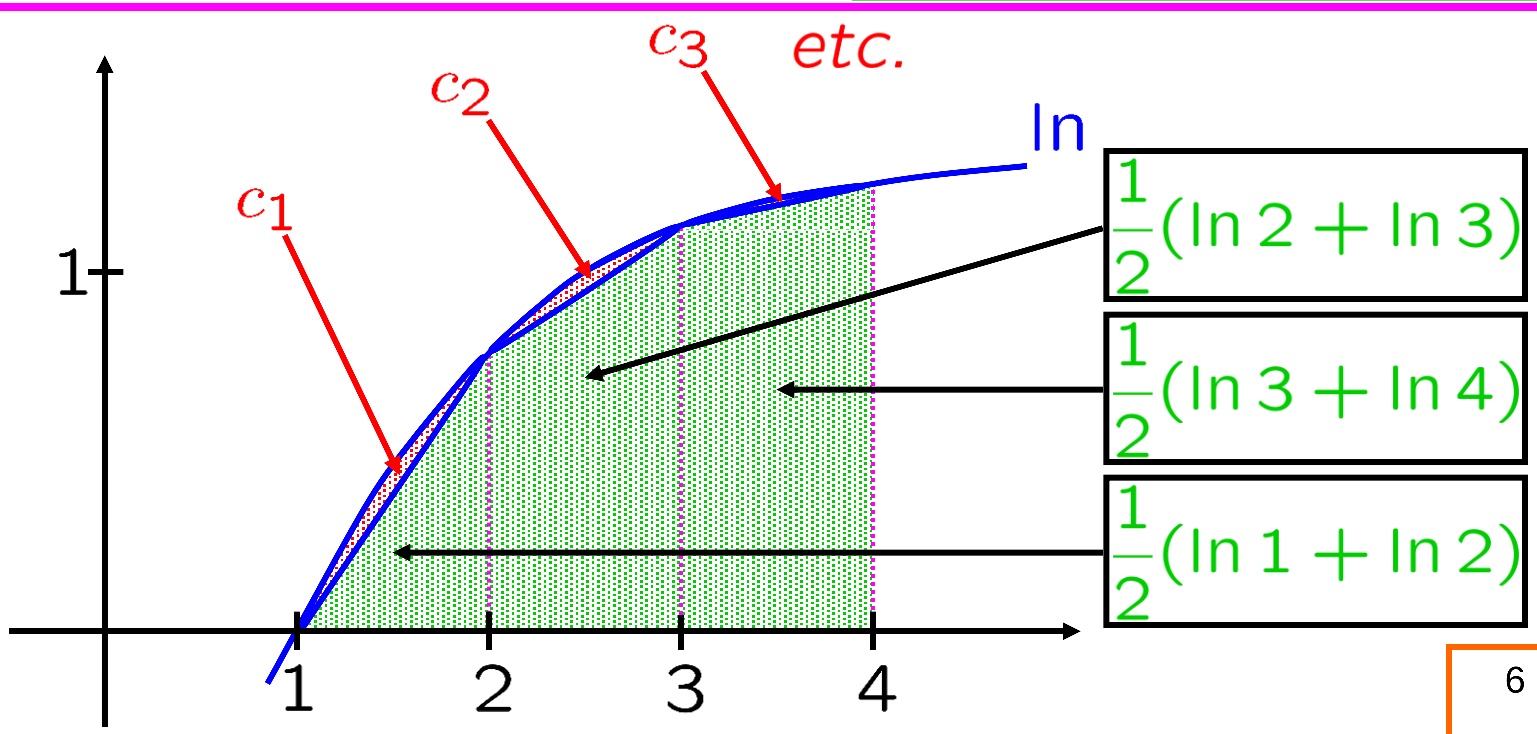
Goal: Asymptotics of $n!$



$$\int_1^4 \ln x\, dx = \ln 1 + \ln 2 + \ln 3 + \ln 4 - \frac{\ln 4}{2} + c_1 + c_2 + c_3$$

$$= \ln 1 + \ln 2 + \ln 3 + \ln 4 - \frac{\ln 4}{2} + c_1 + c_2 + c_3$$

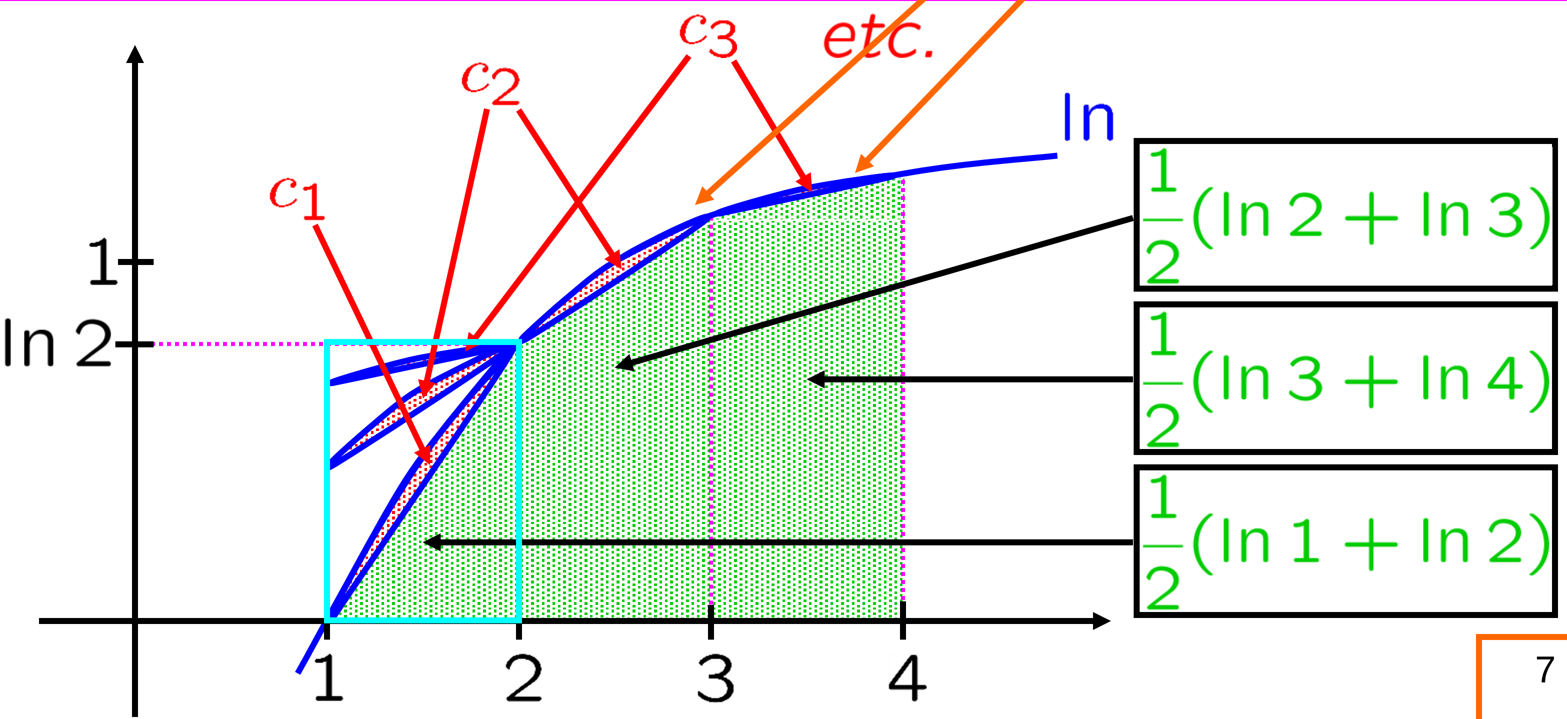
$\boxed{\text{Goal: Asymptotics of } n!}$



$$\int_1^4 \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \ln 4 - \frac{\ln 4}{2} + c_1 + c_2 + c_3$$

$$\int_1^n \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2} + c_1 + c_2 + \cdots + c_n$$

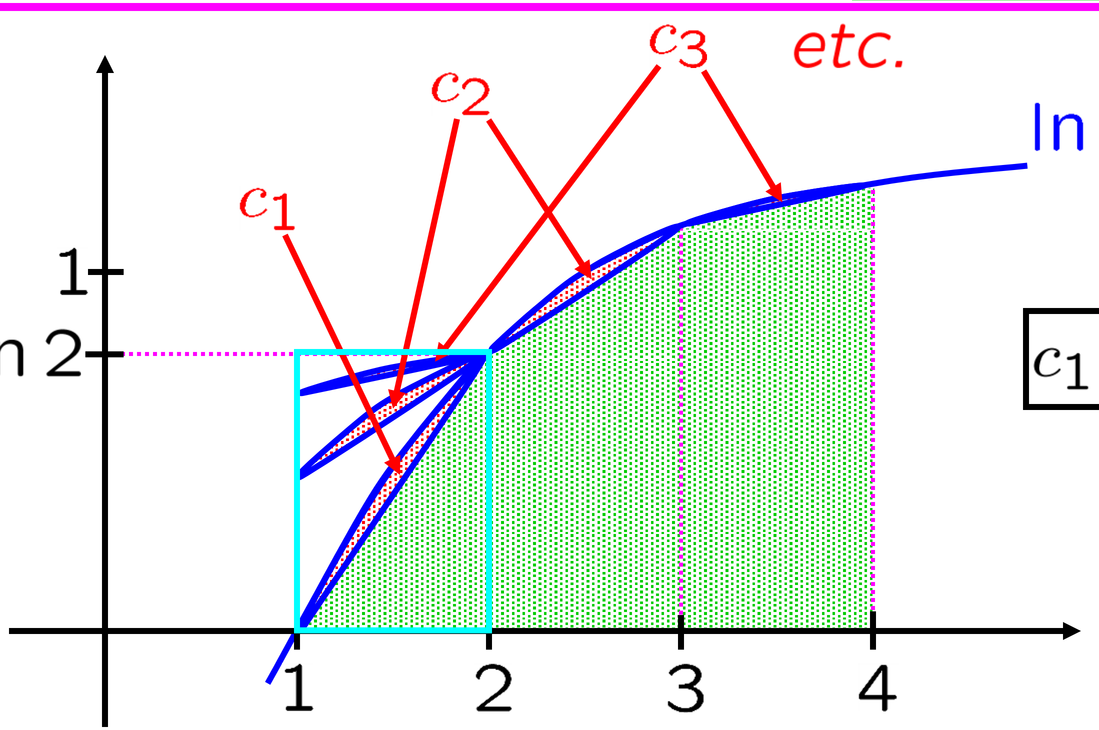
Goal: Asymptotics of $n!$



$$\int_1^4 \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \ln 4 - \frac{\ln 4}{2}$$
$$+c_1 + c_2 + c_3$$

$$\int_1^n \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2}$$
$$+c_1 + c_2 + \cdots + c_n$$

Goal: Asymptotics of $n!$

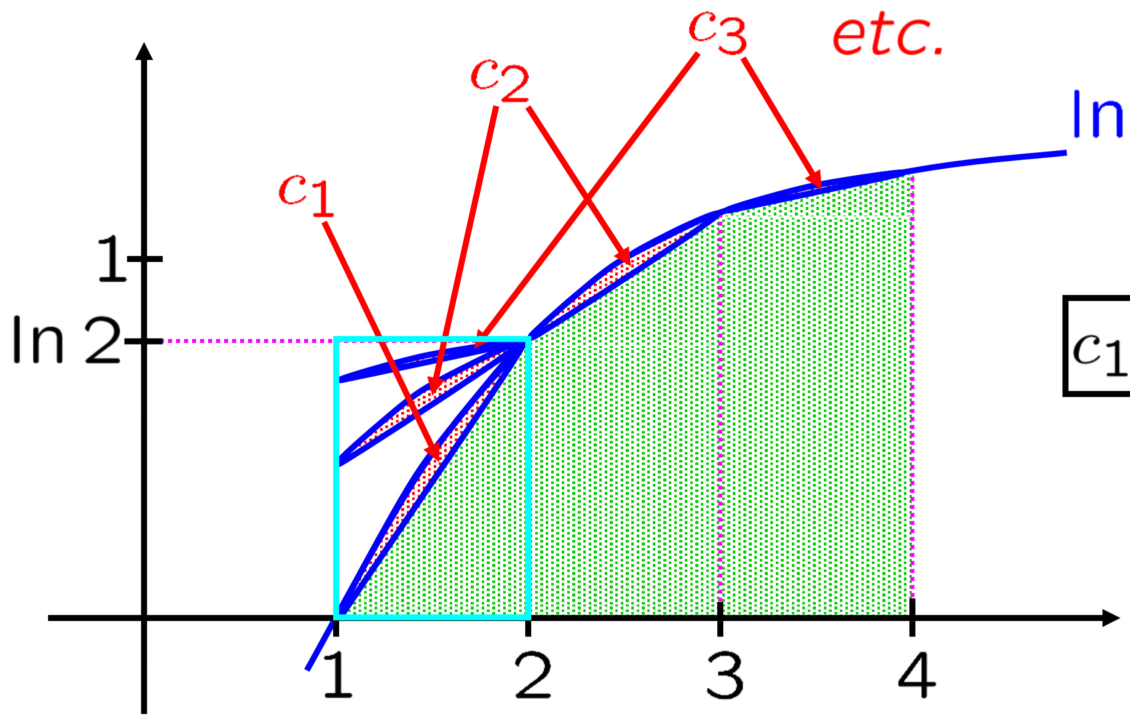


$$\boxed{c_1 + c_2 + c_3 \le \ln 2}$$

$$\int_1^4 \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \ln 4 - \frac{\ln 4}{2} + c_1 + c_2 + c_3$$

$$\int_1^n \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2} + c_1 + c_2 + \cdots + c_n$$

$\boxed{c_1 + c_2 + c_3 + \cdots \text{ converges}}$ $\boxed{c_1 + c_2 + \cdots + c_n \leq \ln 2}$

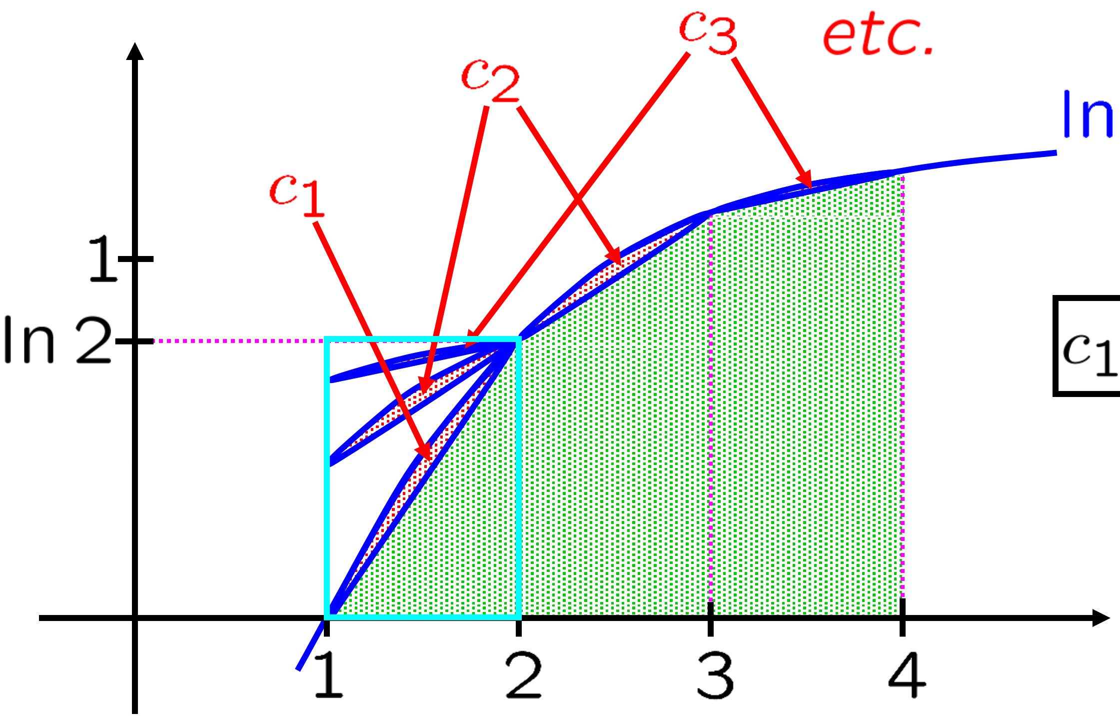


$\boxed{c_1 + c_2 + c_3 \leq \ln 2}$

$$\int \ln x\,dx = \int \underbrace{[\ln x]}\overbrace{[1]}\,dx = \boxed{[\ln x][x]} - \int \boxed{[1/x][x]}\,dx$$
$$= \boxed{x[\ln x]} - x + C$$

$$\int_1^n \ln x\,dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2}$$
$$+ c_1 + c_2 + \cdots + c_n$$

$\boxed{c_1 + c_2 + c_3 + \cdots \text{ converges}}$ $\boxed{c_1 + c_2 + \cdots + c_n \le \ln 2}$



$\boxed{c_1 + c_2 + c_3 \le \ln 2}$

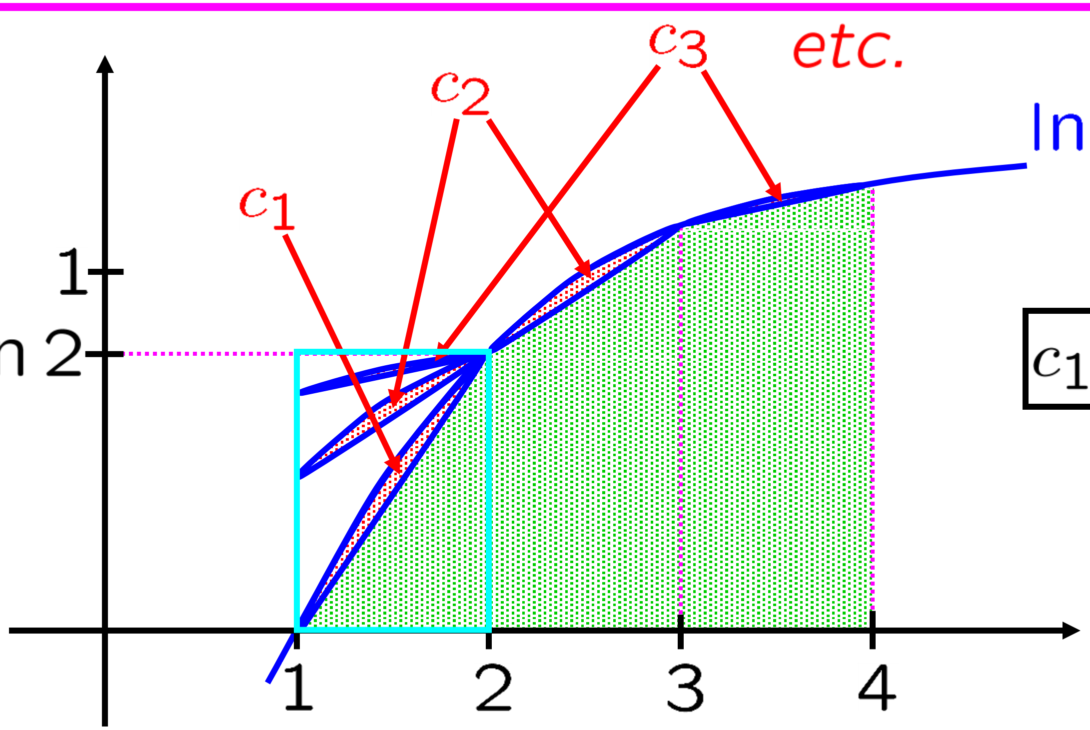
$$\int \ln x\, dx = x[\ln x] - x + C$$

$$= x[\ln x] - x + C$$

$$\int_1^n \ln x\, dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2}$$

$$+ c_1 + c_2 + \cdots + c_n$$

$\boxed{c_1 + c_2 + c_3 + \cdots \text{ converges}}$ $\boxed{c_1 + c_2 + \cdots + c_n \leq \ln 2}$



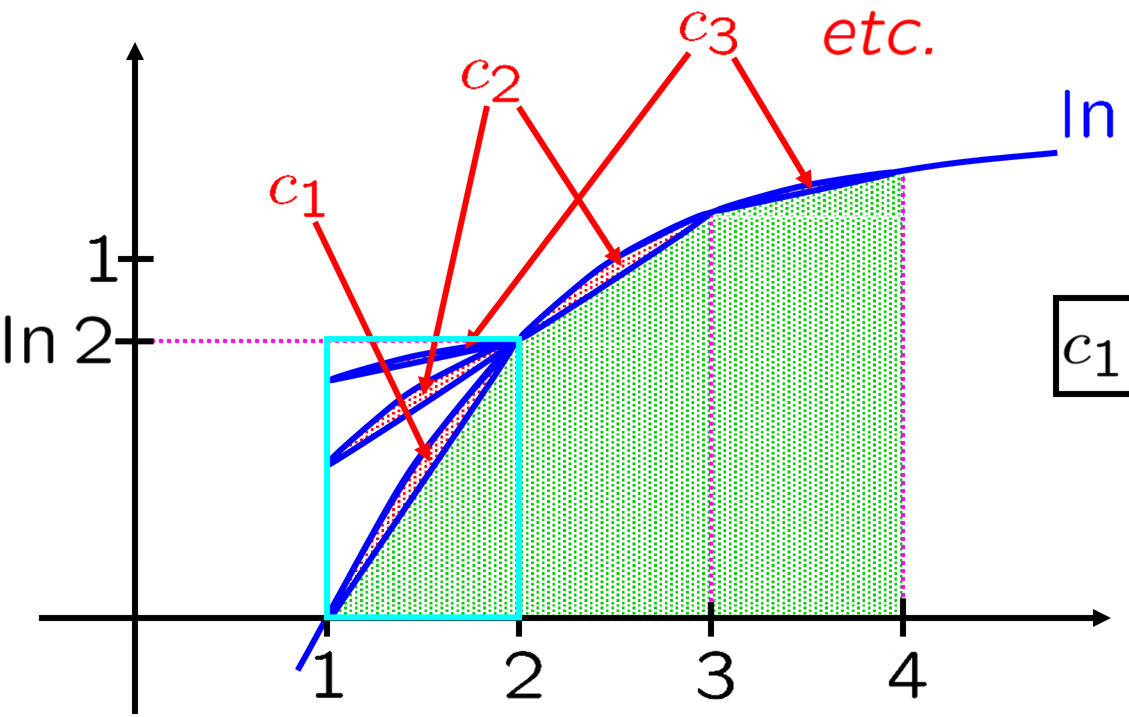
$\boxed{c_1 + c_2 + c_3 \leq \ln 2}$

$$\int \ln x\, dx = x[\ln x] - x + C$$

$$\boxed{\int_1^n \ln x\, dx} = \Big[\boxed{x}[\ln \boxed{x}] - \boxed{x}\Big]_{x:\to \boxed{1}}^{x:\to \boxed{n}} = \boxed{(\boxed{n}[\ln \boxed{n}] - \boxed{n}) - (\boxed{1}[\ln \boxed{1}] - \boxed{1})}$$

$$\int_1^n \ln x\, dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2}$$
$$+ c_1 + c_2 + \cdots + c_n$$

$\boxed{c_1 + c_2 + c_3 + \cdots \text{ converges}}$ $\boxed{c_1 + c_2 + \cdots + c_n \leq \ln 2}$



$\boxed{c_1 + c_2 + c_3 \leq \ln 2}$

$$\int \ln x \, dx \;=\; x[\ln x] - x + C$$

$$\Big[x[\ln x] - x\Big]_{x:\to 1}^{x:\to n} = (n[\ln n] - n) - (1[\ln 1] - 1)$$

$$\int_1^n \ln x \, dx = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{\ln n}{2}$$

$$+ c_1 + c_2 + \cdots + c_n$$

$$c_1 + c_2 + c_3 + \cdots \text{ converges} \qquad c_1 + c_2 + \cdots + c_n \leq \ln 2$$

$$(n[\ln n] - n) + (1[\cancel{\ln 1}] + 1) = [\ln(1 \cdot 2 \cdots n)] - [\ln(\sqrt{n})]$$

$$+ c_1 + c_2 + \cdots + c_n$$

$$(1 - c_1 - \cdots - c_n) + [\ln(n^n)] - n + [\ln(\sqrt{n})] = \ln(n!)$$

$$K_n := e^{1 - c_1 - \cdots - c_n} \qquad K_n [n^n][e^{-n}][\sqrt{n}] = n!$$

exponentiate

$$\boxed{\rho := c_1 + c_2 + c_3 + \cdots}$$

$$\boxed{c_1 + c_2 + c_3 + \cdots \text{ converges}} \quad \boxed{c_1 + c_2 + \cdots + c_n \le \ln 2}$$

$$(n[\ln n] - n) + (1[\ln 1] + 1) = [\ln(1 \cdot 2 \cdots n)] - [\ln(\sqrt{n})] + c_1 + c_2 + \cdots + c_n$$

$$(1 - c_1 - \cdots - c_n) + [\ln(n^n)] - n + [\ln(\sqrt{n})] = \ln(n!)$$

$$K_n := e^{1 - c_1 - \cdots - c_n} \qquad \boxed{K_n[n^n][e^{-n}][\sqrt{n}] = n!}$$

$$K_n \to K := e^{1-\rho}$$

$$K_n \sim K$$

$$K_n \boxed{[n^n][e^{-n}][\sqrt{n}\,]} \sim K \boxed{[n^n][e^{-n}][\sqrt{n}\,]}$$

$a_n \sim b_n$ means
$a_n / b_n \to 1$,
as $n \to \infty$

$$\boxed{\rho := c_1 + c_2 + c_3 + \cdots}$$

$$K := e^{1-\rho}$$

$$K_n[n^n][e^{-n}][\sqrt{n}] = n!$$

$$K := e^{1-\rho}$$

$$K_n[n^n][e^{-n}][\sqrt{n}] = n!$$

$$\|$$

$$K_n[n^n][e^{-n}][\sqrt{n}] \sim K[n^n][e^{-n}][\sqrt{n}]$$

$a_n \sim b_n$ means
$a_n/b_n \to 1$,
as $n \to \infty$

IOU: $K = \sqrt{2\pi}$

$$n! \sim \boxed{K[\sqrt{n}]}\boxed{[n^n][e^{-n}]}$$

?

Stirling's Formula: $n! \sim \boxed{\sqrt{2\pi n}}\boxed{(n/e)^n}$

Goal: Asymptotics of $n!$ ☺

$$\boxed{\rho := c_1 + c_2 + c_3 + \cdots}$$

$$K := e^{1-\rho}$$

$$K_n[n^n][e^{-n}][\sqrt{n}] = n!$$

$$||$$

$$K_n[n^n][e^{-n}][\sqrt{n}] \sim K[n^n][e^{-n}][\sqrt{n}]$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

IOU: $K = \sqrt{2\pi}$

Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

$$\boxed{\text{Stirling's Formula: } n! \sim \sqrt{2\pi n}\,(n/e)^n}$$

$$\boxed{\rho := c_1 + c_2 + c_3 + \cdots}$$

$$K := e^{1-\rho}$$

$$K_n[n^n][e^{-n}][\sqrt{n}] = n!$$

$$||$$

$$K_n[n^n][e^{-n}][\sqrt{n}] \sim K[n^n][e^{-n}][\sqrt{n}]$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$\rho := c_1 + c_2 + c_3 + \cdots$$

$$K = e^{1-\rho}$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \underbrace{[\sin^{n-1} x]}\overbrace{[\sin x]} \, dx$$

$$= \left[[\sin^{n-1} x][-\cos x] \right]_{\boxed{x=0}}^{\boxed{x=\pi/2}} -$$

$$\int_0^{\pi/2} \boxed{[n-1]} [\sin^{n-2} x][\cos x][-\cos x] \, dx$$

$$= (0-0) - \boxed{[n-1]} \int_0^{\pi/2} [\sin^{n-2} x][-\cos^2 x] \, dx$$

$$= [n-1] \int_0^{\pi/2} [\sin^{n-2} x][\boxed{+\cos^2 x}] \, dx$$

$a_n \sim b_n$ means
$a_n/b_n \to 1$,
as $n \to \infty$

$\underline{\text{IOU}}$: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} [\sin^{n-1} x][\sin x]\, dx$$

$$= \Big[[\sin^{n-1} x][-\cos x]\Big]_{x=0}^{x=\pi/2} - \int_0^{\pi/2} [n-1][\sin^{n-2} x][\cos x][-\cos x]\, dx$$

$$= (0-0) - [n-1] \int_0^{\pi/2} [\sin^{n-2} x][-\cos^2 x]\, dx$$

$$= [n-1] \int_0^{\pi/2} [\sin^{n-2} x][\boxed{+\cos^2 x}]\, dx$$

$$= \underline{[n-1]} \int_0^{\pi/2} [\sin^{\boxed{n-2}} x][\boxed{1} - \boxed{\sin^{\boxed{2}} x}]\, dx$$

$$= \underline{[n-1]} \boxed{\left[\int_0^{\pi/2} \sin^{n-2} x\, dx\right]} - \underline{[n-1]} \boxed{\left[\int_0^{\pi/2} \sin^{\boxed{n}} x\, dx\right]}$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$= [n-1]\left[\int_0^{\pi/2} \sin^{n-2} x \, dx\right] - [n-1]\boxed{\left[\int_0^{\pi/2} \sin^n x \, dx\right]}$$

$$= [n-1]\left[\int_0^{\pi/2} \sin^{n-2} x \, dx\right] - [n-1]\left[\int_0^{\pi/2} \sin^n x \, dx\right]$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$\boxed{\underset{n-2}{I_n} := \int_0^{\pi/2} \sin^{\overset{n-2}{n}} x\, dx}$$

$$= [n-1]\overbrace{\boxed{\left[\int_0^{\pi/2} \sin^{n-2} x\, dx\right]}}^{I_{n-2}} - [n-1]\overbrace{\boxed{\left[\int_0^{\pi/2} \sin^{n} x\, dx\right]}}^{I_n}$$

$$I_n = [n-1]\boxed{[I_{n-2}]} - \boxed{[n-1]}\boxed{[I_n]}$$

$$\boxed{[n-1][I_n]} + I_n = [n-1][I_{n-2}]$$

$$\|$$

$$[n][I_n]$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$= [n-1]\left[\int_0^{\pi/2} \sin^{n-2} x \, dx\right] - [n-1]\left[\int_0^{\pi/2} \sin^n x \, dx\right]$$

$$I_n = [n-1][I_{n-2}] - [n-1][I_n]$$

$$[\boxed{n}][I_n] = [n-1][I_{n-2}]$$

$$[n][I_n]$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$= [n-1]\left[\int_0^{\pi/2} \sin^{n-2} x \, dx\right] - [n-1]\left[\int_0^{\pi/2} \sin^n x \, dx\right]$$

$$I_n = [n-1][I_{n-2}] - [n-1][I_n]$$

$$[\boxed{n}][I_n] = [n-1][I_{n-2}]$$

$$I_n = \left[\frac{n-1}{\boxed{n}}\right][I_{n-2}]$$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_n = \left[\frac{n-1}{n}\right] [I_{n-2}]$$

$$\frac{n-1}{n} \to 1$$

$$I_n = \left[\frac{n-1}{n}\right] [I_{n-2}]$$

IOU: $K = \sqrt{2\pi}$

Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_n = \left[\frac{n-1}{n}\right][I_{n-2}]$$

$$\frac{n-1}{n} \to 1$$

$$\frac{n-1}{n} \sim 1$$

MULTIPLY BY I_{n-2}

$$\left[\frac{n-1}{n}\right][I_{n-2}] \sim I_{n-2}$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x\, dx$$

$$I_n = \left[\frac{n-1}{n}\right][I_{n-2}]$$

$$\|$$

$$\left[\frac{n-1}{n}\right][I_{n-2}] \sim I_{n-2}$$

$$\frac{n-1}{n} \to 1$$

$$\frac{n-1}{n} \sim 1$$

$$I_n \sim I_{n-2}$$

$$\left[\frac{n-1}{n}\right][I_{n-2}] \sim I_{n-2}$$

$a_n \sim b_n$ means

$a_n/b_n \to 1$,

as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_n = \left[\frac{n-1}{n}\right][I_{n-2}]$$

$$\|$$

$$\left[\frac{n-1}{n}\right][I_{n-2}] \sim I_{n-2}$$

$0 \le \sin x \le 1$

$0 \le a \le 1 \Rightarrow \cdots \le a^3 \le a^2 \le a \le a$

$$\boxed{\sin^n x \le \sin^{n-1} x \le \sin^{n-2} x}$$

INTEGRATE
FROM 0 TO $\pi/2$

$$\boxed{I_n} \sim \boxed{I_{n-2}}$$

$$\boxed{I_n} \le \boxed{I_{n-1}} \le \boxed{I_{n-2}}$$

$$\boxed{I_n} \sim \boxed{I_{n-1}} \sim \boxed{I_{n-2}}$$

$n :\to n+1$

$$I_{n+1} \sim I_n$$

$n :\to 2n$

$$\boxed{I_{2n+1}} \sim \boxed{I_{2n}}$$

$$\boxed{I_{2n}} \sim \boxed{I_{2n+1}}$$

$a_n \sim b_n$ means

$a_n/b_n \to 1$,
as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x\, dx$$

$$I_n = \left[\frac{n-1}{n}\right][I_{n-2}]$$

$$I_0 = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$

$$I_{2n} \sim I_{2n+1}$$

$$\boxed{I_0 = \frac{\pi}{2}}$$

$$I_{2n} \sim I_{2n+1}$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_n = \boxed{\left[\frac{n-1}{n}\right]} \boxed{[I_{n-2}]}$$

$$I_{2n} \sim I_{2n+1}$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx$$

$$\boxed{I_0 = \frac{\pi}{2}}$$

$$= [-\cos x]_{x=0}^{x=\pi/2}$$

$$= [-0] - [-1]$$

$$= 1$$

$$\boxed{I_1 = 1}$$

$a_n \sim b_n$ means
$a_n/b_n \to 1$,
as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx \qquad I_n = \boxed{\left[\frac{n-1}{n}\right]} \boxed{[I_{n-2}]}$$

$$I_{2n} \sim I_{2n+1}$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx$$

$$\boxed{I_0 = \frac{\pi}{2}}$$

$$= [-\cos x]_{x=0}^{x=\pi/2}$$

$$\boxed{I_1 = 1}$$

$$= [-0] - [-1]$$

$$= 1$$

$$I_n = \boxed{[I_{n-2}]} \boxed{\left[\frac{n-1}{n}\right]}$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_{2n} \sim I_{2n+1}$$

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

$$I_2 = \frac{\pi}{2}\frac{1}{2}$$

$$\frac{\pi}{2}$$

$$I_2 = [I_{2-2}]\left[\frac{2-1}{2}\right] = [I_0]\left[\frac{1}{2}\right]$$

$$I_n = [I_{n-2}]\left[\frac{n-1}{n}\right]$$

IOU: $K = \sqrt{2\pi}$

Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$I_{2n} \sim I_{2n+1}$$

$$I_0 = \frac{\pi}{2} \qquad\qquad I_1 = 1$$

$$I_2 = \frac{\pi}{2}\frac{1}{2} \qquad\qquad I_3 = \frac{2}{3}$$

$$I_3 = [I_{3-2}]\left[\frac{3-1}{3}\right] = \overset{1}{[I_1]}\left[\frac{2}{3}\right]$$

$$I_n = [I_{n-2}]\left[\frac{n-1}{n}\right]$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

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$$I_4 = \frac{\pi}{2}\frac{1}{2}\frac{3}{4}$$

$$\frac{\pi}{2}\frac{1}{2}$$

$$I_4 = [I_{4-2}]\left[\frac{4-1}{4}\right] = [I_2]\left[\frac{3}{4}\right]$$

$$I_n = [I_{n-2}]\left[\frac{n-1}{n}\right]$$

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$$I_4 = \frac{\pi}{2}\frac{1}{2}\frac{3}{4} \qquad I_5 = \frac{2}{3}\frac{4}{5}$$

$$\frac{2}{3}$$

$$I_5 = [I_{5-2}]\left[\frac{5-1}{5}\right] = [I_3]\left[\frac{4}{5}\right]$$

$$I_n = [I_{n-2}]\left[\frac{n-1}{n}\right]$$

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$$I_4 = \frac{\pi}{2}\frac{1}{2}\frac{3}{4} \qquad I_5 = \frac{2}{3}\frac{4}{5}$$

$$I_6 = \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6} \qquad I_7 = \frac{2}{3}\frac{4}{5}\frac{6}{7}$$

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$$I_{\boxed{8}} = \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{\boxed{7}}{\boxed{8}} \qquad I_9 = \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}$$

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$$\boxed{I_{2n} \sim I_{2n+1}}$$

$$I_{\boxed{2n}} = \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{\boxed{2n-1}}{\boxed{2n}}$$

$$I_{\boxed{2n+1}} = \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\cdots\frac{\boxed{2n}}{\boxed{2n+1}}$$

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$$\boxed{I_{2n} \sim I_{2n+1}}$$

$$I_{2n} = \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2n-1}{2n} \qquad I_{2n+1} = \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\cdots\frac{2n}{2n+1}$$

$$\frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2n-1}{2n} \sim \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\cdots\frac{2n}{2n+1}$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

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$$\frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2n-1}{2n}\underbrace{\boxed{\frac{2}{2}\frac{4}{4}\frac{6}{6}\frac{8}{8}\cdots\frac{2n}{2n}}}_{1}\sim\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\cdots\frac{2n}{2n+1}$$

$$\frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2n-1}{2n} \qquad \sim \qquad \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\cdots\frac{2n}{2n+1}$$

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Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$\boxed{\frac{\pi}{2}}\,\frac{\boxed{1\,3\,5\,7}}{\boxed{2\,4\,6\,8}}\cdots\frac{\boxed{2n-1}}{\boxed{2n}}\;\frac{\boxed{2\,4\,6\,8}}{\boxed{2\,4\,6\,8}}\cdots\frac{\boxed{2n}}{\boxed{2n}} \sim \frac{2\,4\,6\,8}{3\,5\,7\,9}\cdots\frac{2n}{2n+1}\,\underbrace{\boxed{\frac{2\,4\,6\,8}{2\,4\,6\,8}\cdots\frac{2n}{2n}}}_{1}$$

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot\cdots\cdot(2n-1)\cdot(2n) = (2n)!$$

$$\boxed{\frac{\pi}{2}}\,\frac{\boxed{(2n)!}}{\boxed{[2\cdot 4\cdot\cdots\cdot(2n)]^2}}$$

$$[2\cdot 4\cdot 6\cdot 8\cdot\cdots\cdot(2n)]^2$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$\frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\cdots\frac{2n-1}{2n}\frac{2}{2}\frac{4}{4}\frac{6}{6}\frac{8}{8}\cdots\frac{2n}{2n} \sim \frac{\boxed{2\,4\,6\,8 \quad 2n}}{\boxed{3\,5\,7\,9 \quad 2n+1}}\cdots\frac{\boxed{2\,4\,6\,8 \quad 2n}}{\boxed{2\,4\,6\,8 \quad 2n}}$$

$$[\,2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2n)\,]^2$$

$$\frac{\pi}{2}\frac{(2n)!}{[2 \cdot 4 \cdot \cdots \cdot (2n)]^2} \sim \frac{\boxed{[2 \cdot 4 \cdot \cdots \cdot (2n)]^2}}{\boxed{(2n+1)!}}$$

$$(2n+1)!$$

$$\|$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot \cdots \cdot (2n) \cdot (2n+1)$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$$\frac{\pi}{2}\frac{(2n)!}{\boxed{[2\cdot 4\cdot\cdots\cdot(2n)]^{\boxed{2}}}} \sim \frac{\boxed{[2\cdot 4\cdot\cdots\cdot(2n)]^{\boxed{2}}}}{(2n+1)!}$$

$$\frac{\pi}{2}\frac{(2n)!}{[2\cdot 4\cdot\cdots\cdot(2n)]^2} \sim \frac{[2\cdot 4\cdot\cdots\cdot(2n)]^2}{(2n+1)!}$$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}$, $\rho := c_1 + c_2 + c_3 + \cdots$

$$\frac{\pi}{2}\frac{(2n)!}{[2\cdot 4\cdot\cdots\cdot(2n)]^2} \sim \frac{[2\cdot 4\cdot\cdots\cdot(2n)]^2}{(2n+1)!}$$

$$\frac{\pi}{2}[(2n)!]\,[(2n+1)!] \sim [2\cdot 4\cdot\cdots\cdot(2n)]^4$$

$$/\!/$$

$$\frac{\pi}{2}[(2n)!][(2n)!][2n+1]$$

$$||$$

$$\frac{\pi}{2}[(2n)!]^2[2n+1]$$

LEADING TERM

$2n + 1$

$(2n+1)!$

$||$

$[(2n)!][2n+1]$

$$\wr \longleftrightarrow \wr$$

$$\frac{\pi}{\cancel{2}}[(2n)!]^2[\cancel{2}n]$$

$2n$

$$||$$

$$\pi[(2n)!]^2[n]$$

$a_n \sim b_n$ means $a_n/b_n \to 1$, as $n \to \infty$

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$$\frac{\pi}{2}\frac{(2n)!}{[2\cdot 4\cdot \cdots \cdot (2n)]^2} \sim \frac{[2\cdot 4\cdot \cdots \cdot (2n)]^2}{(2n+1)!}$$

$$\frac{\pi}{2}[(2n)!][(2n+1)!] \sim [2\cdot 4\cdot \cdots \cdot (2n)]^{\boxed{4}}$$

$$\frac{\pi}{2}[(2n)!][(2n)!][2n+1] \qquad [(2^n)(1\cdot 2\cdot \cdots \cdot n)]^{\boxed{4}}$$

$$\frac{\pi}{2}[(2n)!]^2[2n+1] \qquad [(2^n)(n!)]^{\boxed{4}}$$

$$\frac{\pi}{\cancel{2}}[(2n)!]^2[\cancel{2}n] \qquad 2^{4n}[n!]^4$$

$$\pi[(2n)!]^2[n]$$

$a_n \sim b_n$ means

$a_n/b_n \to 1,$
as $n \to \infty$

IOU: $K = \sqrt{2\pi}$ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

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$$\pi[(2n)!]^2[n] \sim 2^{4n}[n!]^4$$

$$\sim$$

$$2^{4n}[n!]^4$$

$$\pi[(2n)!]^2[n]$$

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as $n \to \infty$

Know: $K = \sqrt{2\pi}$ ☺ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

Know: $K = e^{1-\rho}, \quad \rho := c_1 + c_2 + c_3 + \cdots$

$n :\to 2n$

$$(\boxed{2n})! \sim K[\sqrt{\boxed{2n}}][(\boxed{2n})^{\boxed{2n}}][e^{-\boxed{2n}}] \qquad \boxed{n}! \sim K[\sqrt{\boxed{n}}][\boxed{n}^{\boxed{n}}][e^{-\boxed{n}}]$$

$$\pi[(2n)!]^2[n] \sim 2^{4n}[n!]^4$$

$$\pi[K[\sqrt{2n}][(2n)^{2n}][e^{-2n}]]^{\boxed{2}}[n] \sim 2^{4n}[K[\sqrt{n}][n^n][e^{-n}]]^{\boxed{4}}$$

$$\pi\boxed{K^2}[2\cancel{n}][(2n)^{\boxed{4n}}]\cancel{[e^{-4n}]}[\cancel{n}] \sim 2^{4n}\boxed{K^4}\cancel{[n^2]}[n^{4n}]\cancel{[e^{-4n}]}$$

$$\pi \quad 2 \quad \cancel{2^{4n}}[\cancel{n^{4n}}] \sim \cancel{2^{4n}}\boxed{K^2} \quad [\cancel{n^{4n}}]$$

$$2\pi \sim K^2$$

$$2\pi = K^2$$

$$K = e^{1-\rho} > 0$$

$$\sqrt{2\pi} = K$$

$a_n \sim b_n$ means

$a_n/b_n \to 1,$

as $n \to \infty$

Know: $K = \sqrt{2\pi}$ ☺ Know: $n! \sim K[\sqrt{n}][n^n][e^{-n}]$

$$\boxed{\text{Stirling's Formula: } n! \sim \sqrt{2\pi n}\,(n/e)^n}$$

PROVED!

STOP