Financial Mathematics

From Stirling's Formula to the Central Limit Theorem

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

Def'n: Say, $\forall n$, $a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n/b_n \to 1$.

$$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3$$

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$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3 \leftarrow n :\to n^2 + 2n$ $\qquad n^2 + 2n \to \infty$

$n :\to q_n$

Fact: $x_n \sim y_n,\ q_n \to \infty \quad \Rightarrow \quad x_{q_n} \sim y_{q_n}$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$ $\longleftarrow$ $n :\to n^2 + 2n$

$$(n^2+2n)! \sim \sqrt{2\pi(n^2+2n)}\,((n^2+2n)/e)^{n^2+2n}$$

Def'n: Say, $\forall n,\ a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n/b_n \to 1$.

$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3$ $\longleftarrow$ $n :\to n^2 + 2n$ $\qquad n^2 + 2n \to \infty$

$\frac{1}{2}(n^2+2n)^3 + 500(n^2+2n)^2 \sim \frac{1}{2}(n^2+2n)^3$

Fact: $x_n \sim y_n,\ q_n \to \infty \quad \Rightarrow \quad x_{q_n} \sim y_{q_n}$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads, T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \le \text{ht} \le 69 + 5$?

$$69 - 5 \le 69 + 5\frac{H-T}{\sqrt{N}} \le 69 + 5$$

$$-5 \le 5\frac{H-T}{\sqrt{N}} \le 5$$

DIVIDE BY 5

$$-1 \le \frac{H-T}{\sqrt{N}} \le 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads
T tails

Male height (inches): $69 + 5\dfrac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

||

Probability that: $-1 \leq \dfrac{H-T}{\sqrt{N}} \leq 1$?

$$-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads, T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$ square root

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

||

Probability that: $-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$? Answer: $\approx 68\%$

Grav accel (ft/sec^2): $32 + 10^6\frac{H-T}{N}$ NO square root

< 0.000000000001

Probability that: $32 - \frac{10^6}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^6}{\sqrt{N}}$?

||

Probability that: $-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$? Answer: $\approx 68\%$

Applied Coin-Flipping

$N = 10^{10^{100}}$ $\qquad$ N coin flips $\qquad$ H heads, T tails

< 0.000000000001

Probability that: $32 - \dfrac{10^8}{\sqrt{N}} \leq \text{acc} \leq 32 + \dfrac{10^8}{\sqrt{N}}$?

||

Probability that: $-100 \leq \dfrac{H-T}{\sqrt{N}} \leq 100$?

Answer > 99.999999999999%

Grav accel (ft/sec^2): $32 + 10^6 \dfrac{H-T}{N}$ NO square root

< 0.000000000001

Probability that: $32 - \dfrac{10^6}{\sqrt{N}} \leq \text{acc} \leq 32 + \dfrac{10^6}{\sqrt{N}}$?

||

Probability that: $-1 \leq \dfrac{H-T}{\sqrt{N}} \leq 1$?

Answer: $\approx$ 68%

Applied Coin-Flipping

$N = 10^{10^{100}}$ $\quad$ N coin flips $\quad$ H heads, T tails

< 0.000000000001

Probability that: $32 - \frac{10^8}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^8}{\sqrt{N}}$?

$\parallel$

Probability that: $-100 \leq \frac{H-T}{\sqrt{N}} \leq 100$?

Answer $> 99.999999999999\%$

Grav accel (ft/sec^2): $32 + 10^6 \frac{H-T}{N}$ NO square root

Maybe all physical phenomena is probabilistic, **but** sometimes the denominator is so large that we can't detect it.

We'll see that $\sqrt{N}$ is the "right" denom. for detectibly probabilistic phenomena.

Coin-Flipping Applied to Finance

$N = 10^{10^{100}}$

Current stock price: 1 USD

30 days from now, derivative contract pays:
5 USD if $1/e <$ price $< e$,   2 USD otherwise
$e \approx 2.71828$

Expected payout?

Grav accel (ft/sec^2): $32 + 10^6 \frac{H - T}{N}$ NO square root

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Maybe all physical phenomena is probabilistic, but sometimes the denominator is so large that we can't detect it.

We'll see that $\sqrt{N}$ is the "right" denom. for detectibly probabilistic phenomena.

Coin-Flipping Applied to Finance

$N = 10^{10^{100}}$ split-second $:= \dfrac{30 \text{ days}}{N}$

Current stock price: 1 USD
ln $= 0$

30 days from now, derivative contract pays:
5 USD if $1/e <$ price $< e$, 2 USD otherwise
$-1 \leq$ ln ≤ 1

Expected payout? Answer: $\approx 5(0.68) + 2(0.32)$

Approx. Black-Scholes

Model: Each split-second, ln(price)
either increases or decreases by $1/\sqrt{N}$,
with 50% chance of uptick,
50% chance of downtick.

ln(ending price)

Probability that: $-1 \leq \dfrac{H - T}{\sqrt{N}} \leq 1$? Why? Answer: $\approx$ 68%

Coin-Flipping Applied to Finance

Choose an integer $n \geq 1$. *e.g.*: $n = N/2$

ln(stock price) starts at 0 on a (horizontal) number line.

Flip a fair coin $2n$ times in 30 days.

With each head, it
moves $1/\sqrt{2n}$ units in the positive direction (right).

With each tail, it
moves $1/\sqrt{2n}$ units in the negative direction (left).

---

e.g.: $n = 9$
$2n = 18$

Probability that: $-1 \leq \dfrac{H-T}{\sqrt{N}} \leq 1$? Why?

Answer: $\approx 68\%$

Coin-Flipping Applied to Finance

Choose an integer $n \geq 1$.

ln(stock price) starts at 0 on a (horizontal) number line.

Flip a fair coin $2n$ times in 30 days.

With each head, it
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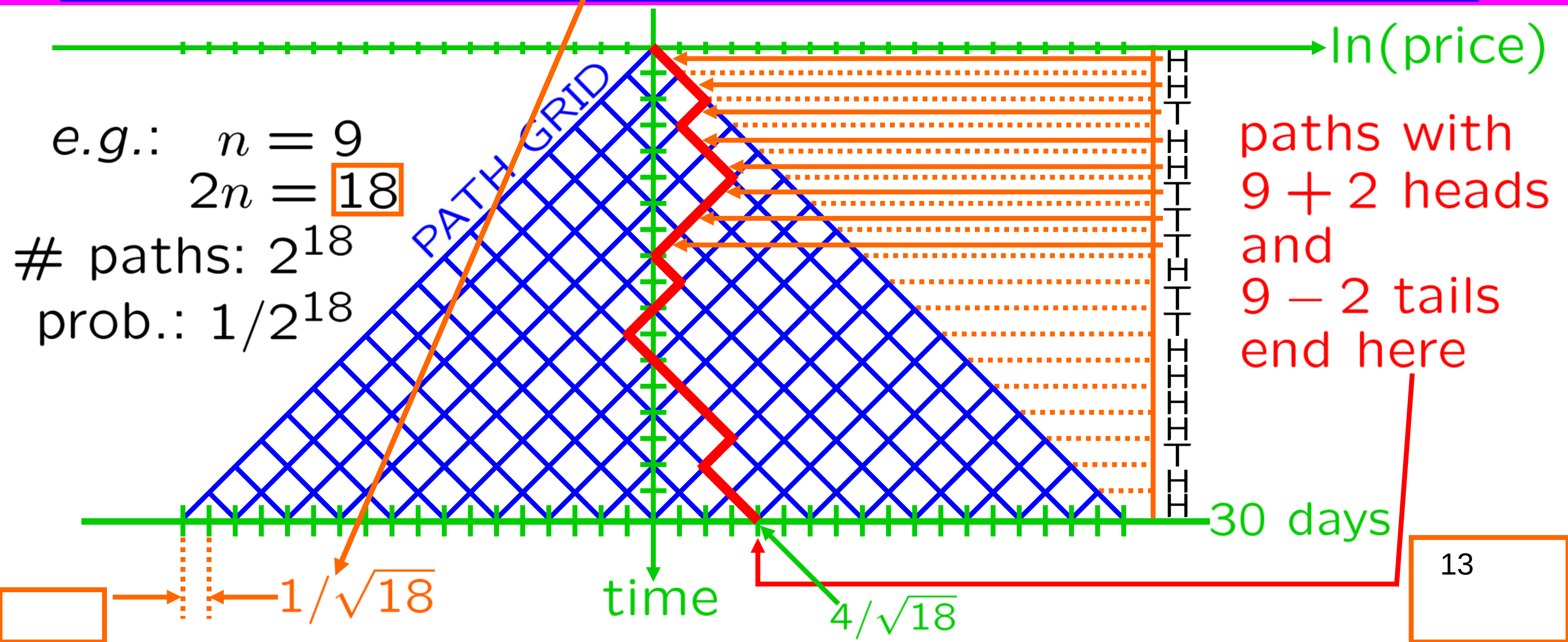
PATH



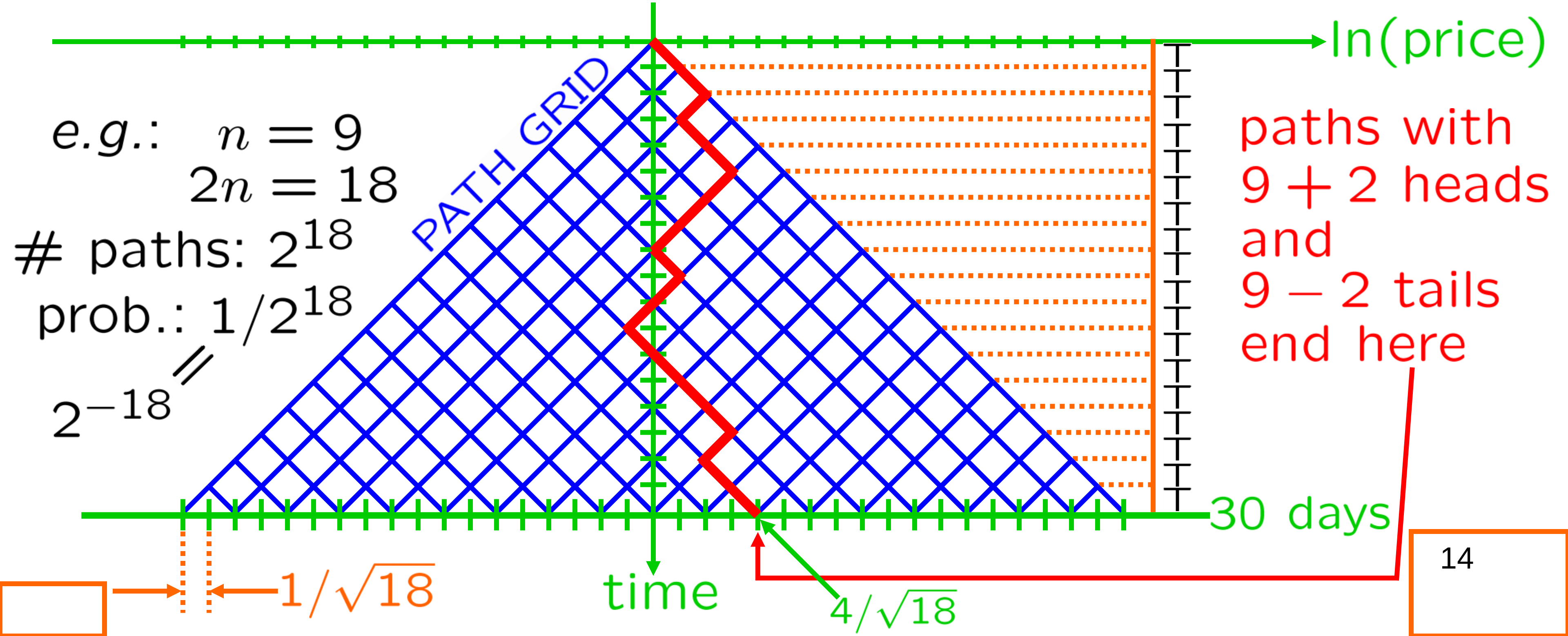
Question: What is the probability of ending at $4/\sqrt{18}$?

Answer: $2^{-18} \times$ # of paths ending at $4/\sqrt{18}$.

# of collections of 18 Hs and Ts with $9+2$ Hs and $9-2$ Ts

$$\binom{18}{9+2}$$

Start with 18 Ts, choose $9+2$ of them and change the chosen $9+2$ to Hs.



Build a historgram
(a.k.a. "bar graph")

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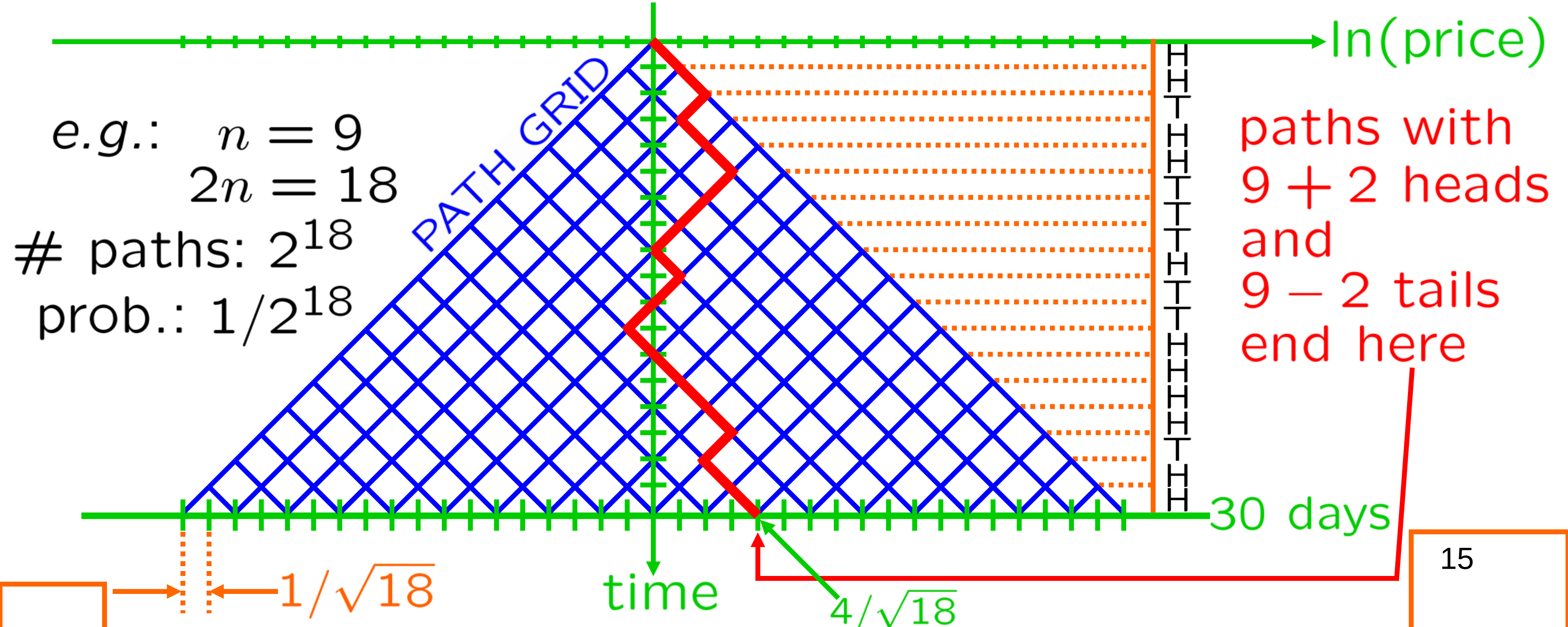
Answer: $2^{-18} \times \binom{18}{9+2}$



Build a historgram (a.k.a. "bar graph"): Start by placing a rectangle ("bar") horizontally centered at $4/\sqrt{18}$

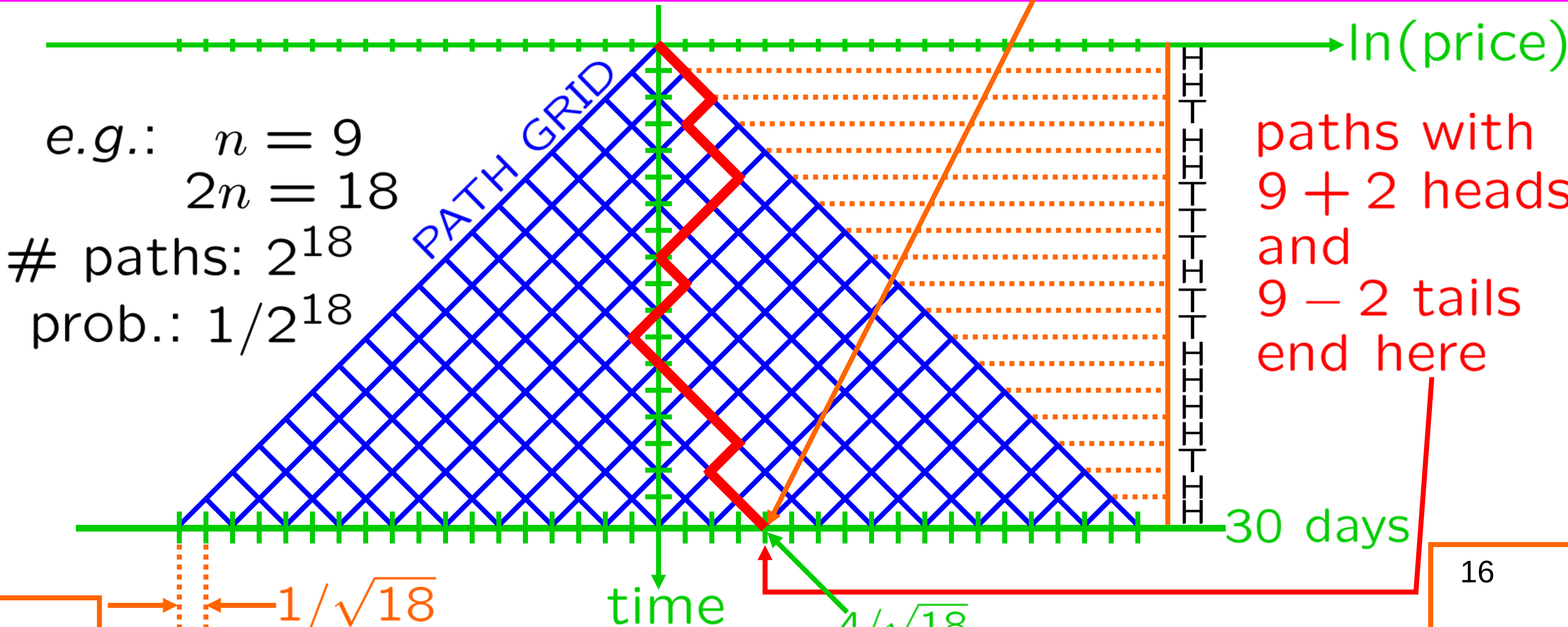
height? whose area is $\dfrac{1}{2^{18}}\dbinom{18}{9+2}$.

width?

next bar?

Question: What is the probability of ending at $4/\sqrt{18}$?

Answer: $2^{-18}\times\dbinom{18}{9+2}$



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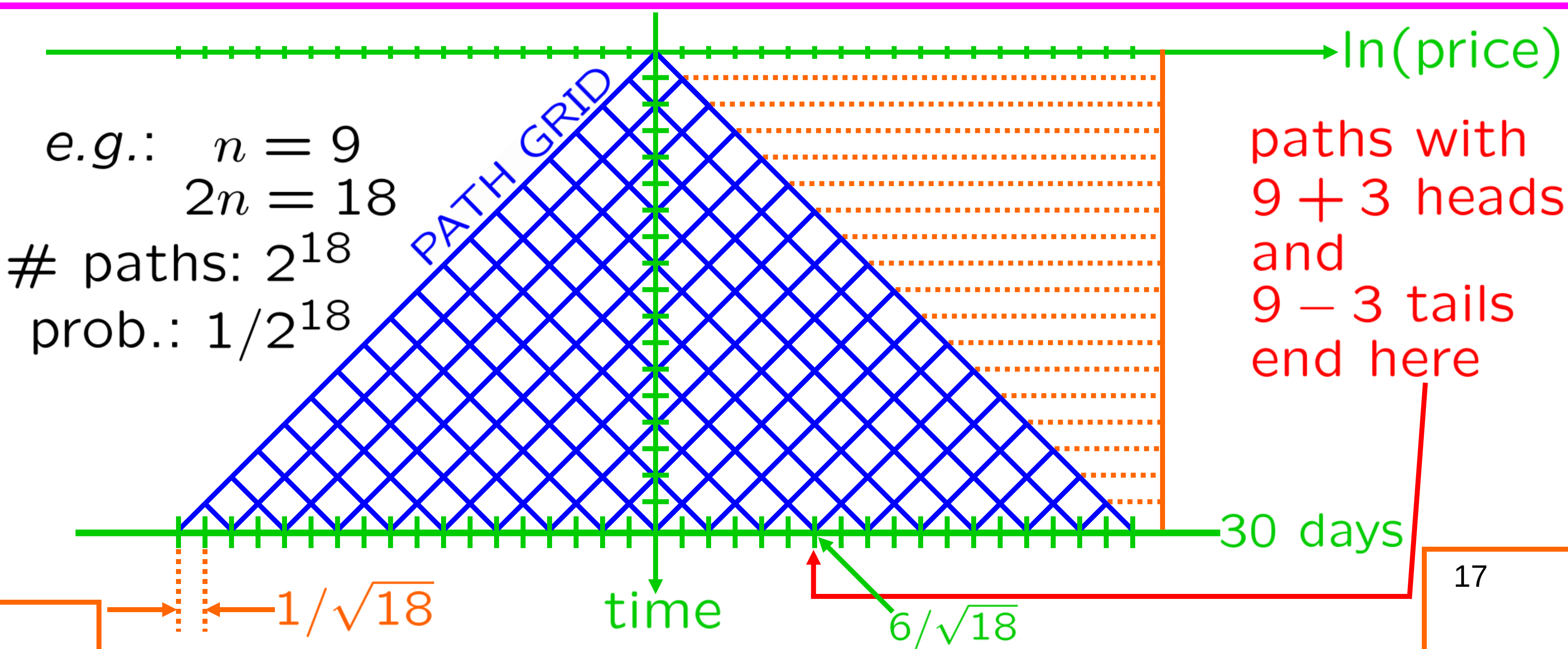

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whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

height? $\frac{\text{area}}{\text{width}}$

width? $2/\sqrt{18}$

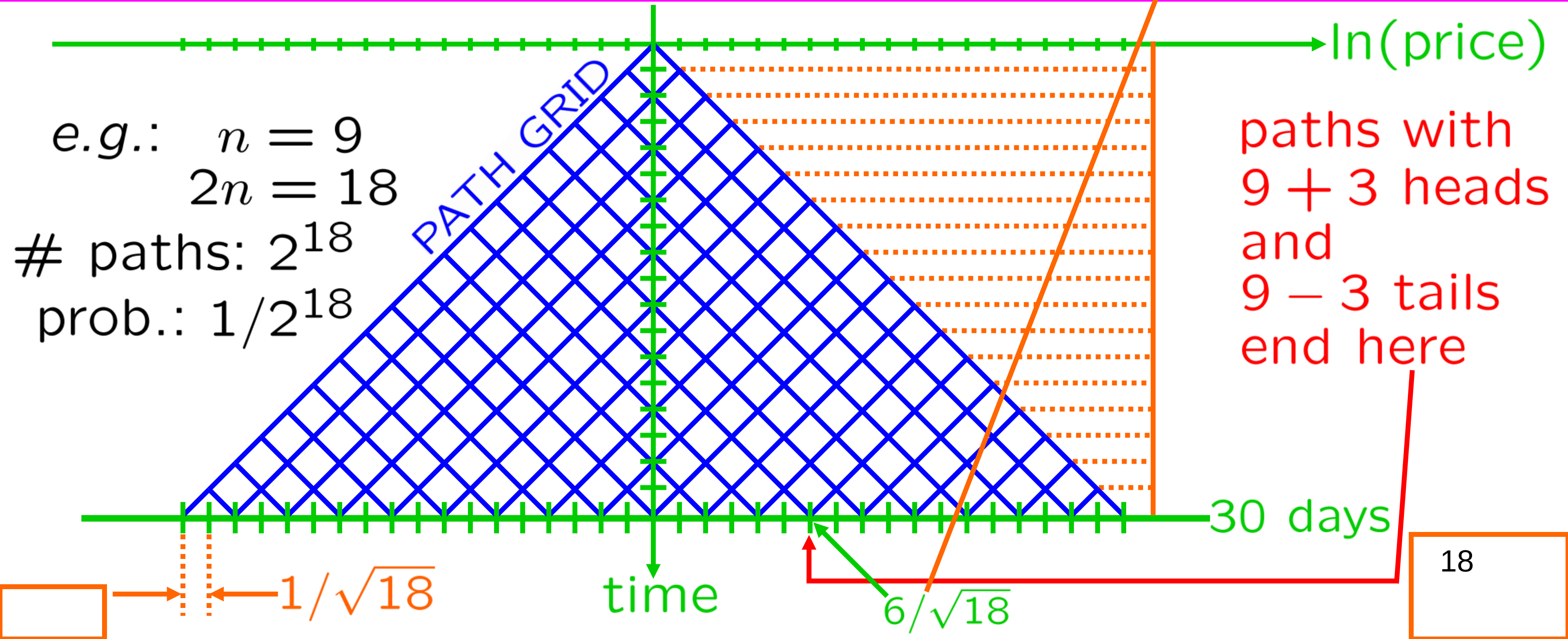
ALL BARS

next bar?

subtract

Next bar: horizontally centered at $6/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3} = 0.07082$$



Build a historgram: Start by placing a rectangle ("bar") (a.k.a. "bar graph")

horizontally centered at $4/\sqrt{18}$

ALL BARS: height? $\frac{\text{area}}{\text{width}}$ width? $2/\sqrt{18}$

whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$. $= 0.12140$

Next bar: horizontally centered at $6/\sqrt{18}$

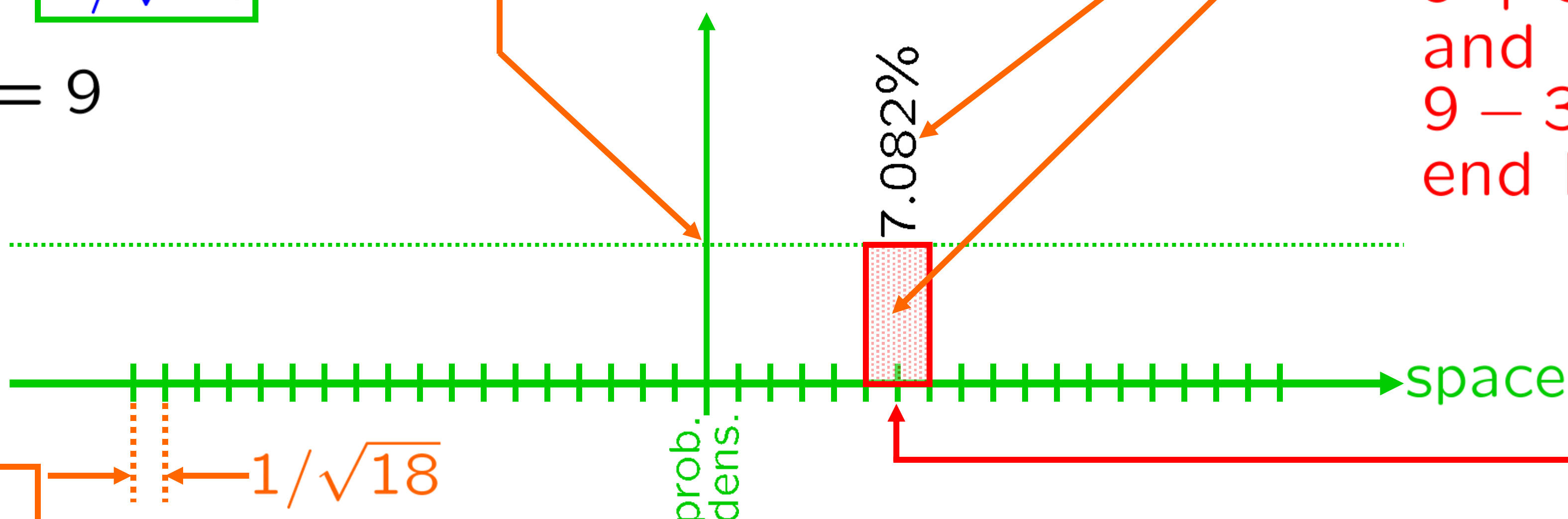
$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3}. = 0.07082$$

Height:

$$\frac{0.07082}{2/\sqrt{18}} = 0.15023$$

$n = 9$

paths with $9 + 3$ heads and $9 - 3$ tails end here



Build a historgram: Start by placing a rectangle ("bar") (a.k.a. "bar graph")

horizontally centered at $4/\sqrt{18}$

height? $\frac{\text{area}}{\text{width}}$ ALL BARS

width? $2/\sqrt{18}$

whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$. $= 0.12140$

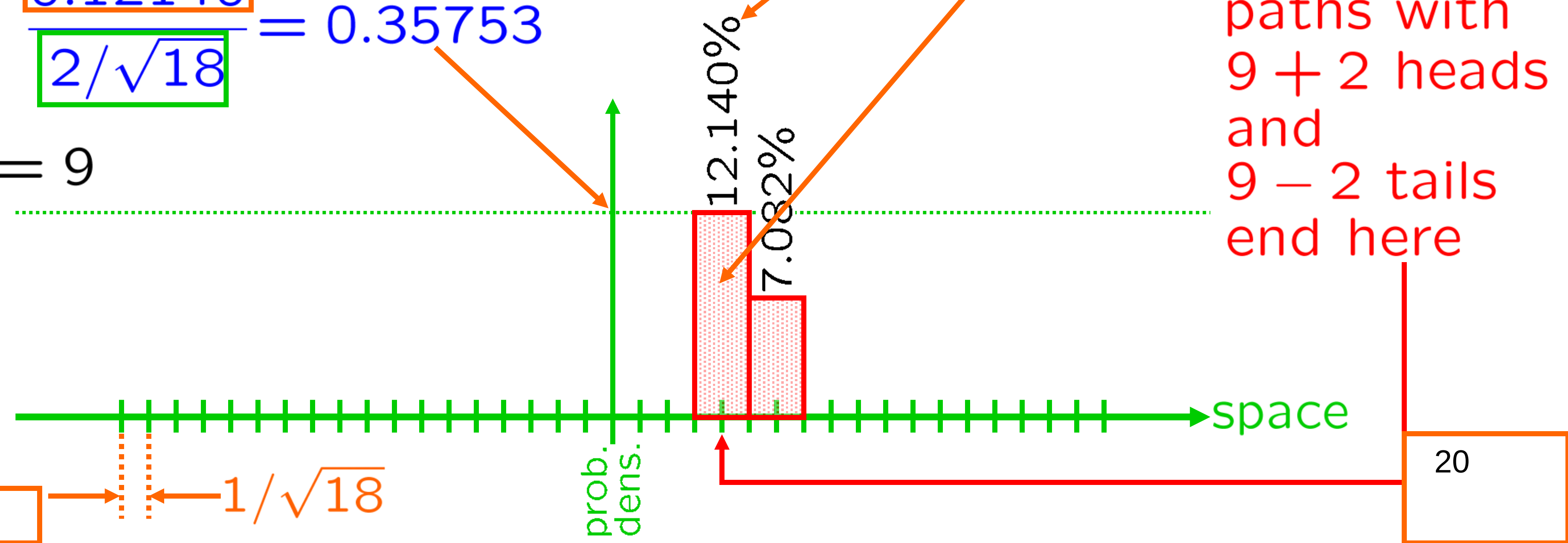
Next bar: horizontally centered at $6/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3}.$$

Height:

$$\frac{0.12140}{2/\sqrt{18}} = 0.35753$$

$n = 9$



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height? $\frac{\text{area}}{\text{width}}$ ALL BARS

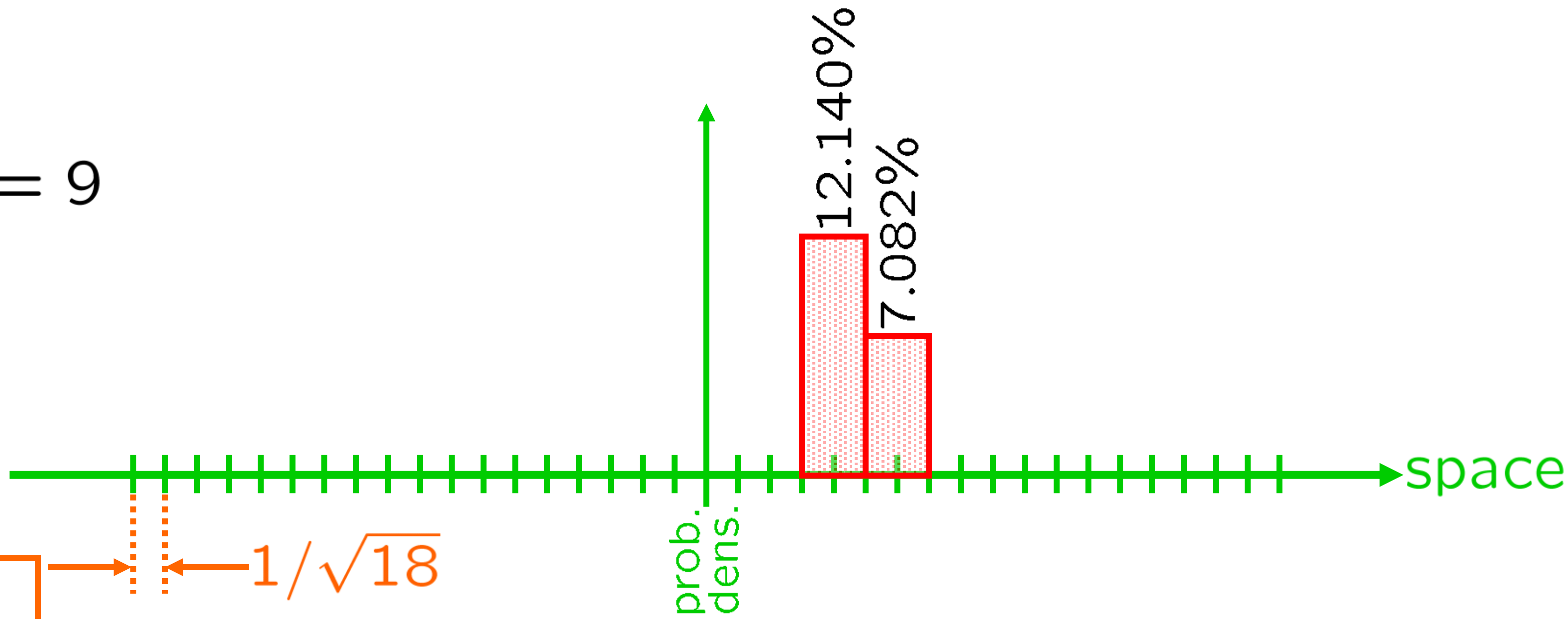
width? $2/\sqrt{18}$

whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

$\forall$integers $k \in [-9, 9]$, make a bar horizontally centered at $(2k)/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+k}.$$

$n = 9$



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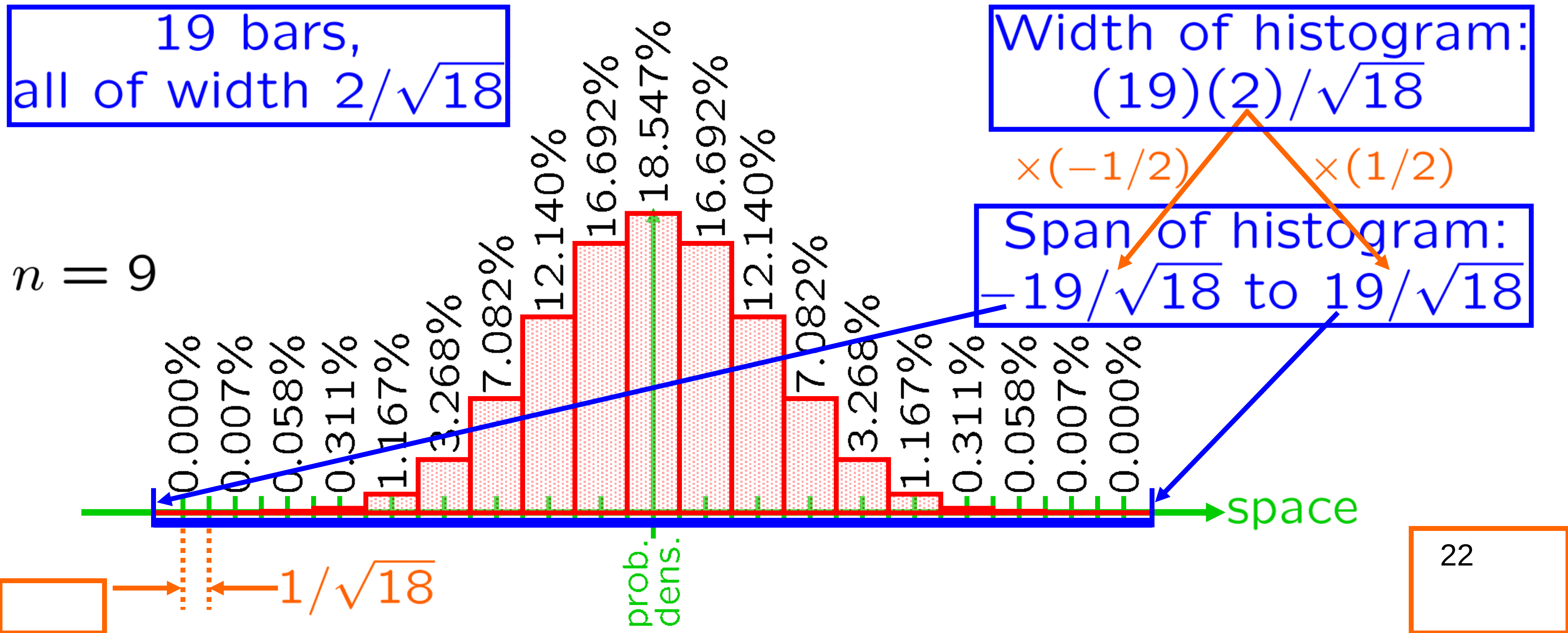
height? $\frac{\text{area}}{\text{width}}$

width? $2/\sqrt{18}$

ALL BARS

$\forall$integers $k \in [-9, 9]$, make a bar horizontally centered at $(2k)/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+k}.$$



Build a sequence of historgrams: $\forall$integers $n \geq 1$,

$\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Width of the bars: $2/\sqrt{2n}$

$\forall$integers $k \in [-9, 9]$, make a bar horizontally centered at $(2k)/\sqrt{18}$

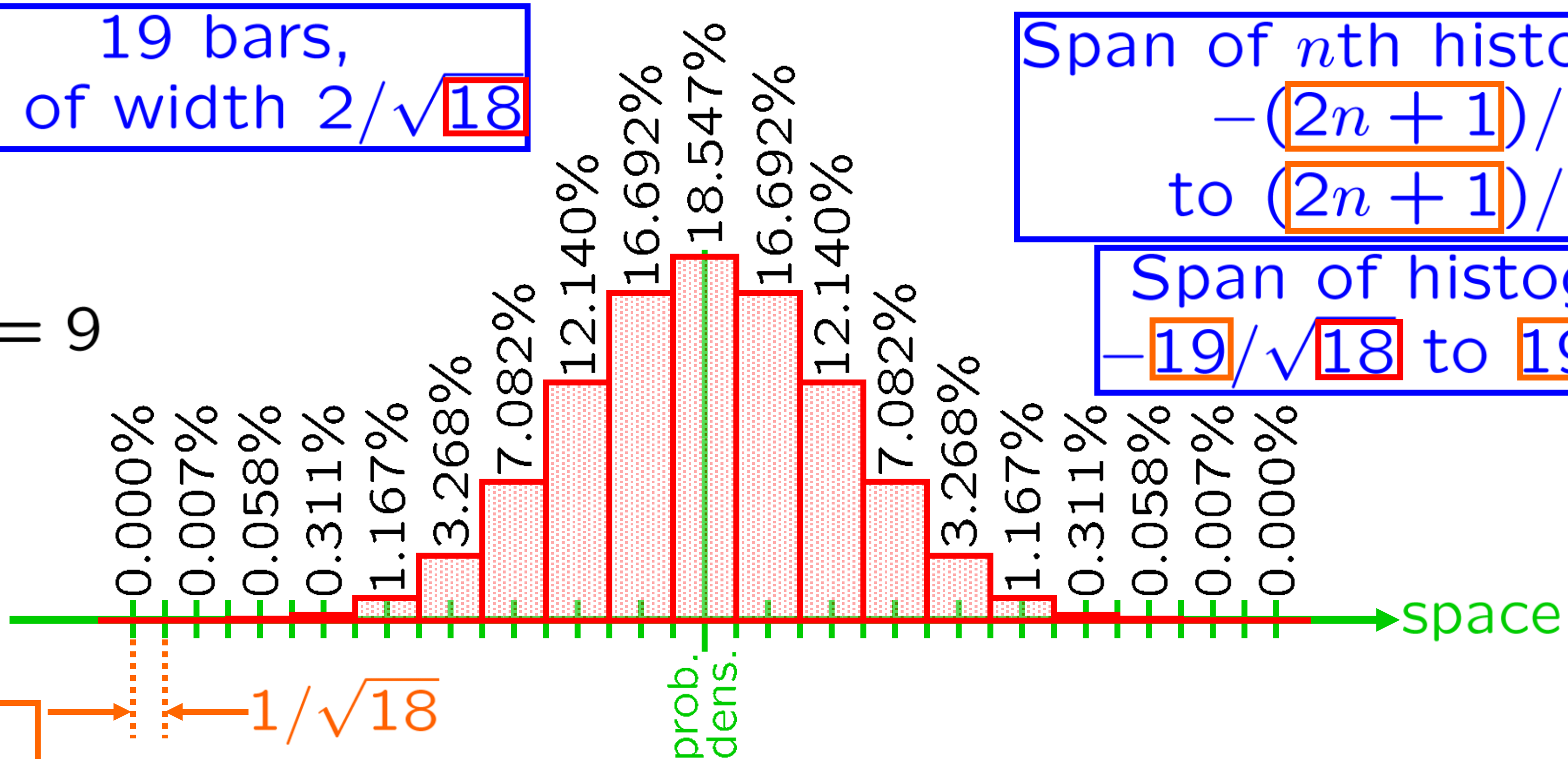
$$\text{area} = \frac{1}{2^{18}} \binom{18}{9+k}.$$

19 bars, all of width $2/\sqrt{18}$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Span of histogram: $-19/\sqrt{18}$ to $19/\sqrt{18}$

$n = 9$



Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

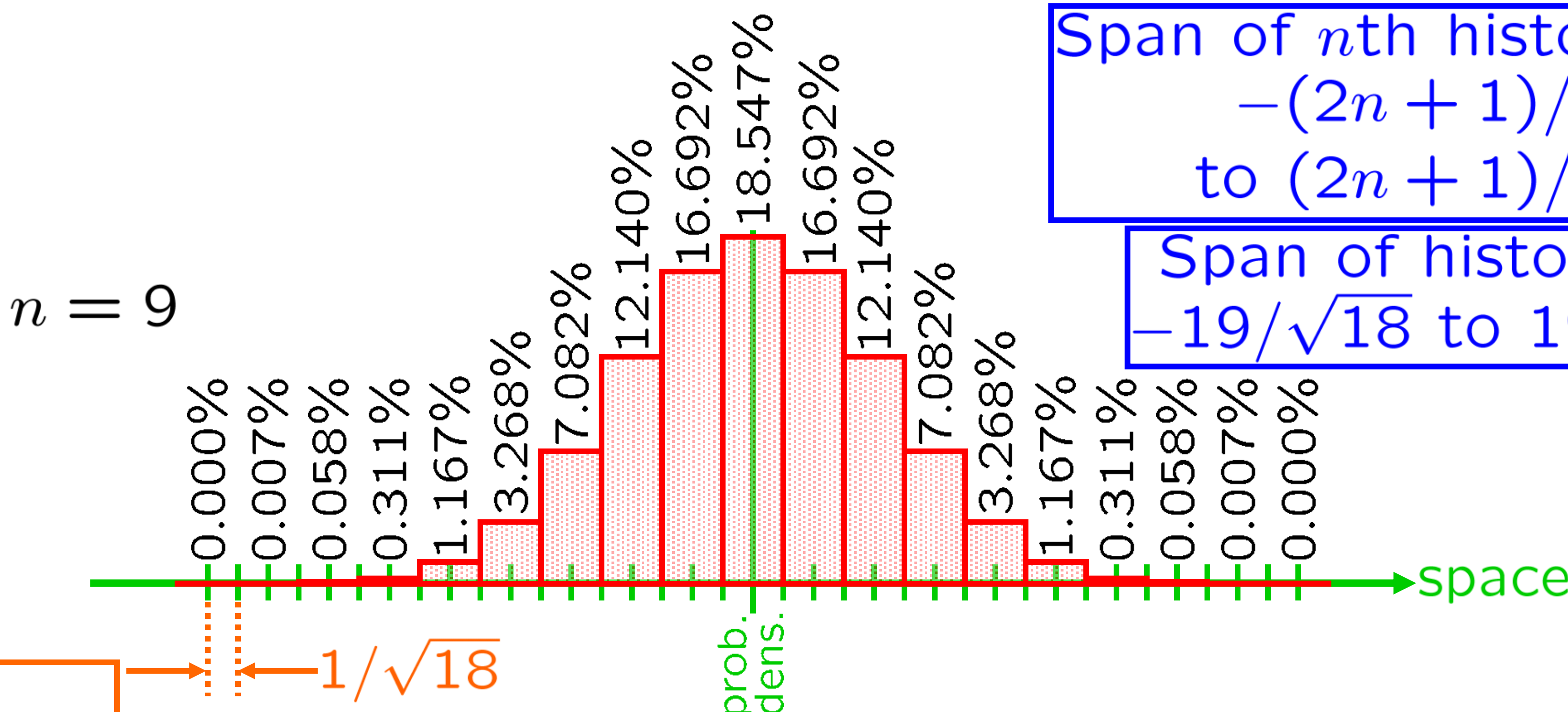
$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Probability that: $-1 \leq \dfrac{H - T}{\sqrt{N}} \leq 1$? $\quad N = 10^{10^{100}}$

Answer: $\left(\begin{array}{c}\text{Area under hist. } \#N/2 \\ \text{between } x = -1 \text{ and } x = 1\end{array}\right) \overset{?}{\approx} 68\%$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Span of histogram: $-19/\sqrt{18}$ to $19/\sqrt{18}$



Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

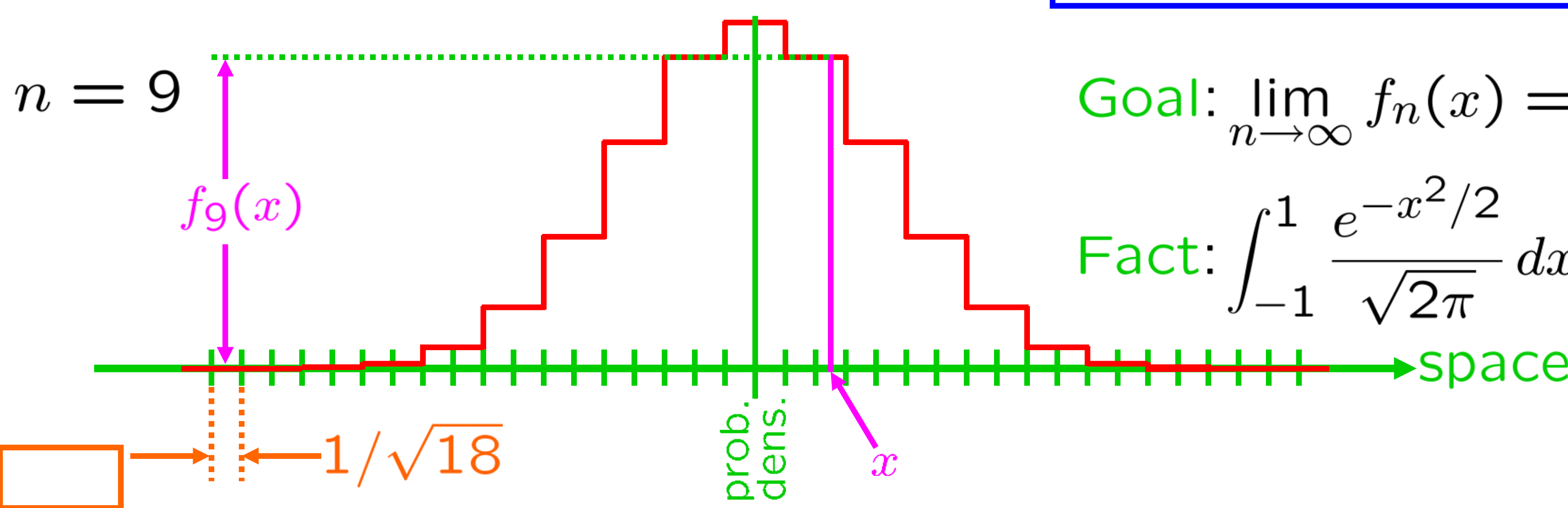
$$\text{area} = \frac{1}{2^{2n}}\binom{2n}{n+k}.$$

Probability that: $-1 \leq \dfrac{H-T}{\sqrt{N}} \leq 1$? $N = 10^{10^{100}}$

Answer: (Area under hist. #$N/2$ between $x=-1$ and $x=1$) $\overset{?}{\approx}$ 68%

Idea: Find $\lim\limits_{n\to\infty} f_n(x)$, then integrate from $x=-1$ to $x=1$.

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$



Goal: $\lim\limits_{n\to\infty} f_n(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

Fact: $\displaystyle\int_{-1}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \approx 0.68$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$

$\Downarrow$?

x in the span of nth histogram

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Goal: $\lim_{n \to \infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$n \geq x^2/2$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$
$\Downarrow$?
x in the span of nth histogram

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Goal: $\lim_{n\to\infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$$n \geq x^2/2 \Leftrightarrow -\sqrt{2n} \leq x \leq \sqrt{2n}$$

$$2n \geq x^2$$

$$x^2 \leq 2n$$ TAKE SQUARE ROOT

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$

$\Downarrow$?

x in the span of nth histogram

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Goal: $\lim_{n\to\infty} f_n(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

$n \geq x^2/2 \Leftrightarrow -\sqrt{2n} \leq x \leq \sqrt{2n} \Rightarrow$ x is in the span of the nth histogram

$-\sqrt{2n} = -(2n)/\sqrt{2n} \geq -(2n+1)/\sqrt{2n}$

$\sqrt{2n} = (2n)/\sqrt{2n} \leq (2n+1)/\sqrt{2n}$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$
$\Downarrow$
x in the span of nth histogram

$k \to k_n$

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

$k \to k_n$

height of kth bar in nth histogram:

$$\frac{1}{2^{2n}} \binom{2n}{n+k} / (2/\sqrt{2n}) = \frac{1}{2^{2n}} \binom{2n}{n+k} \frac{\sqrt{2n}}{2}$$

$k \to k_n$

$\frac{\text{area}}{\text{width}}$

Central Limit Theorem: Let $x \in \mathbb{R}$. $\forall n \geq x^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq x \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$.

$h_n = f_n(x)$

Goal: $\lim_{n\to\infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

Then $h_n \to \frac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Central Limit Theorem: ~~Let $x \in \mathbb{R}$.~~ $\forall n \geq x^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq x \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \dfrac{1}{2^{2n}} \dbinom{2n}{n + k_n} \dfrac{\sqrt{2n}}{2}$.

Then $h_n \rightarrow \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}]$$

$\sqrt{2n}$

$$2k_n - 1 \leq 7\sqrt{2n} \leq 2k_n + 1$$

Central Limit Theorem at 7: $\forall n \geq 7^2/2,$ choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

Asymptotics of k_n:

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}]$$

$$2k_n - 1 \le 7\sqrt{2n} \le 2k_n + 1 \qquad 2k_n - 1 \le 2k_n \le 2k_n + 1$$

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1 \qquad 2k_n - 1 \sim 2k_n \sim 2k_n + 1$$

$$2k_n - 1 \sim 2k_n + 1$$

LEADING TERMS

$$2n + 1 \sim 2n - 1 \qquad n :\to k_n$$

$$k_n \to \infty$$

Central Limit Theorem at 7: $\forall n \ge 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \dfrac{1}{2^{2n}} \dbinom{2n}{n+k_n} \dfrac{\sqrt{2n}}{2}$.

Then $h_n \to \dfrac{e^{-7^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1$$

$$\begin{array}{ccccccc} 2k_n - 1 & \sim & 7\sqrt{2n} & \sim & 2k_n + 1 \\ \parallel & & & & \parallel \\ 2k_n - 1 & \sim & 2k_n & \sim & 2k_n + 1 \end{array}$$

Central Limit Theorem (at 7): $\forall n \geq 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n + k_n} \frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1$$
$$\| \qquad\qquad\qquad //$$
$$2k_n - 1 \sim 2k_n \sim 2k_n + 1$$

divide by 2: $2k_n \sim 7\sqrt{2n}$

$$k_n \sim 7\sqrt{2n}/\boxed{2} = 7\sqrt{\boxed{2}n/\boxed{4}} = 7\sqrt{n/\boxed{2}}$$

$\sqrt{4}$

Central Limit Theorem at 7: $\forall n \geq 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$k_n \sim \qquad 7\sqrt{n/2}$$

Central Limit Theorem at 7: $\forall n \geq 7^2/2$, choose k_n s.t.

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$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

DIVIDE BY n

$$\frac{k_n}{n} \sim \frac{7\sqrt{n/2}}{n} \to 0$$

$$x_n \sim y_n \to z \Rightarrow \frac{x_n}{y_n} \to 1,\ y_n \to z \Rightarrow \left[\frac{x_n}{y_n}\right][y_n] \to [1][z] \Rightarrow x_n \to z$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$$ Want: $$h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \sim \frac{7\sqrt{n/2}}{n} \to 0$$

$$x_n \sim y_n \to z \qquad x_n \sim y_n \to z \Rightarrow x_n \to z \qquad \Rightarrow x_n \to z$$

$$\frac{k_n}{n} \to 0$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$ Want: $$h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

ASYMPTOTIC $$\frac{n+k_n}{n} = 1 + \frac{k_n}{n} \to 1$$

ASYMPTOTIC $$\frac{n-k_n}{n} = 1 - \frac{k_n}{n} \to 1$$

ADD AND SUBTRACT 1

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \dfrac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][((2n)-(n+k_n))!]} \frac{\sqrt{2n}}{2}$$

$\sqrt{4}$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][(2n-n-k_n)!]} \sqrt{\frac{2n}{4}}$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \sqrt{\frac{n}{2}}$$

$$\binom{B}{A} = \frac{B!}{A!(B-A)!}$$

$k_n \sim 7\sqrt{n/2}$, $h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$\frac{(2n)!}{[a_n!]\,[b_n!]}$$

$$\frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$a_n := n + k_n \sim n$

$b_n := n - k_n \sim n$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!]\,[b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sqrt{2\pi(2n)}\,(2n/e)^{2n}$$

$$\sqrt{2\pi a_n}(a_n/e)^{a_n} \qquad \sqrt{2\pi b_n}(b_n/e)^{b_n} \qquad n :\longrightarrow b_n$$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

$$x_n \sim y_n \to z \quad \Rightarrow \quad x_n \to z$$

$$x_n \sim y_n \to \infty \quad \Rightarrow \quad x_n \to \infty$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!][b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sim \frac{1}{2^{2n}} \qquad \frac{\sqrt{2\pi(2n)}(2n/e)^{2n}}{[\sqrt{2\pi a_n}(a_n/e)^{a_n}][\sqrt{2\pi b_n}(b_n/e)^{b_n}]} \qquad \sqrt{\frac{n}{2}}$$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

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$$a_n := n + k_n \sim n \to \infty$$

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$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!][b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sim \frac{1}{2^{2n}} \quad \frac{\sqrt{2\pi(2n)}(2n/e)^{2n}}{[\sqrt{2\pi a_n}(a_n/e)^{a_n}][\sqrt{2\pi b_n}(b_n/e)^{b_n}]} \sqrt{\frac{n}{2}}$$

$$= \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \quad \sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \quad \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

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$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$2^{2n} n^{2n} e^{-2n}$$

$$\sim$$

$$\frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \quad \sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \quad \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

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$$2^{2n} n^{2n} e^{-2n}$$

$$a_n^{a_n} e^{-a_n} \qquad b_n^{b_n} e^{-b_n}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \dfrac{e^{-7^2/2}}{\sqrt{2\pi}}.$

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$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]}\sqrt{\frac{n}{2}} \qquad \frac{1}{2^{2n}}\frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$= \sqrt{\frac{2\pi(2n)}{[2\pi a_n][2\pi b_n]}\frac{n}{2}} \qquad \frac{1}{2^{2n}}\frac{2^{2n}n^{2n}e^{-2n}}{[a_n^{a_n}e^{-a_n}][b_n^{b_n}e^{-b_n}]}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

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$$= \sqrt{\frac{2\pi(2n)}{[2\pi a_n][2\pi b_n]}\frac{n}{2}} \qquad \frac{1}{2^{2n}}\frac{2^{2n}n^{2n}e^{-2n}}{[a_n^{a_n}e^{-a_n}][b_n^{b_n}e^{-b_n}]}$$

$$= \sqrt{\frac{n^2}{[a_n][2\pi b_n]}} \qquad \frac{n^{2n}e^{-2n}}{[a_n^{a_n}e^{-a_n}][b_n^{b_n}e^{-b_n}]}$$

$$= \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \frac{n^{2n}}{a_n^{a_n}b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n}e^{-b_n}}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

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$$\frac{e^{-2n}}{e^{-(a_n+b_n)}}$$

$$\sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \frac{n^{2n}}{a_n^{a_n} b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n} e^{-b_n}}$$

$k_n \sim 7\sqrt{n/2}$, $h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

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$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \frac{n^{2n}}{a_n^{a_n} b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n} e^{-b_n}}$$

$$\frac{n^{2n}}{a_n^{n} a_n^{k_n} b_n^{n} b_n^{-k_n}} \quad \frac{e^{-2n}}{e^{-(a_n+b_n)}}$$

$$\frac{n^{2n}}{a_n^n b_n^n} \quad \frac{b_n^{k_n}}{a_n^{k_n}} \quad \frac{e^{-2n}}{e^{-2n}}$$

$$\left(\frac{n^2}{a_n b_n}\right)^n \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$a_n + b_n = 2n$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\frac{n}{a_n} \to 1$$

$$\left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \dfrac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$
$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \left(\frac{n^2}{a_n b_n}\right)^n \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\sqrt{\frac{n^2}{2\pi a_n b_n}} \longrightarrow \sqrt{\frac{1}{2\pi}}$$

IOU $\left(\frac{n^2}{a_n b_n}\right)^n \longrightarrow e^{7^2/2}$

IOU $\left(\frac{b_n}{a_n}\right)^{k_n} \longrightarrow e^{-7^2}$

$\frac{n}{a_n} \to 1$

$\frac{n}{b_n} \to 1$

MULTIPLY TOGETHER: $\frac{n^2}{a_n b_n} \to 1$

$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$\frac{k_n}{n} \to 0$

$a_n := n + k_n \sim n \to \infty$

$b_n := n - k_n \sim n \to \infty$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \left(\frac{n^2}{a_n b_n}\right)^n \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\downarrow \qquad \text{IOU}\downarrow \qquad \text{IOU}\downarrow$$

$$\sqrt{\frac{1}{2\pi}} \quad e^{7^2/2} \quad e^{-7^2}$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

$$h_n \to \left[\sqrt{\frac{1}{2\pi}}\right]\left[e^{7^2/2}\right]\left[e^{-7^2}\right]$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \left(\frac{n^2}{a_n b_n}\right)^n \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\sqrt{\frac{n^2}{2\pi a_n b_n}} \longrightarrow \sqrt{\frac{1}{2\pi}}$$

IOU: $\left(\frac{n^2}{a_n b_n}\right)^n \longrightarrow e^{7^2/2}$

IOU: $\left(\frac{b_n}{a_n}\right)^{k_n} \longrightarrow e^{-7^2}$

$$(7^2/2) - 7^2 = -7^2/2$$

$$h_n \to \left[\sqrt{\frac{1}{2\pi}}\right]\left[e^{7^2/2}\right]\left[e^{-7^2}\right] = \frac{e^{-7^2/2}}{\sqrt{2\pi}}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$\left(\frac{n^2}{a_n b_n}\right)^{n} \quad \xrightarrow{\text{IOU}} \quad e^{7^2/2}$$

$$\left(\frac{b_n}{a_n}\right)^{k_n} \quad \xrightarrow{\text{IOU}} \quad e^{-7^2}$$

Want: $\left(\frac{b_n}{a_n} - 1\right) k_n \to -7^2$ and $k_n \to \infty$

Want: $\left(\frac{n^2}{a_n b_n} - 1\right) n \to 7^2/2$ and $n \to \infty$ ☺ obvious

IOU: $(s_n - 1) t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$k_n \sim 7\sqrt{n/2} \to \infty$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

Want: $\left(\frac{b_n}{a_n} - 1\right) k_n \to -7^2$ and $k_n \to \infty$ ☺

Want: $\left(\frac{n^2}{a_n b_n} - 1\right) n \to 7^2/2$ and ☺ $n \to \infty$

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$\left(\frac{b_n}{a_n} - 1\right) k_n = \left(\frac{b_n - a_n}{a_n}\right) k_n = \left(\frac{-2k_n}{a_n}\right) k_n = -2\left[\frac{k_n^2}{a_n}\right]$$

$$b_n - a_n = (\not{n} - k_n) - (\not{n} + k_n) = -2k_n$$

Want: $\left(\frac{b_n}{a_n} - 1\right) k_n \to -7^2$ and $k_n \to \infty$ ☺

Want: $\left(\frac{n^2}{a_n b_n} - 1\right) n \to 7^2/2$ and $n \to \infty$ ☺

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n + k_n} \frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$
$$b_n := n - k_n \sim n \to \infty$$

$$\left(\frac{b_n}{a_n}-1\right)k_n = \left(\frac{b_n - a_n}{a_n}\right)k_n = \left(\frac{-2k_n}{a_n}\right)k_n = -2\left[\frac{k_n^2}{a_n}\right]$$

$$\frac{k_n}{7\sqrt{n/2}} \to 1 \qquad \frac{2k_n^2}{7^2 n} = \frac{k_n^2}{7^2(n/2)} \to 1 \qquad \frac{2k_n^2}{7^2 a_n} = \left[\frac{2k_n^2}{7^2 n}\right]\left[\frac{n}{a_n}\right] \to 1$$

SQUARE

MULTIPLY TOGETHER

$$\frac{n}{a_n} \to 1$$

Want: $\left(\frac{b_n}{a_n}-1\right)k_n \to -7^2$ and $k_n \to \infty$ ☺

Want: $\left(\frac{n^2}{a_n b_n}-1\right)n \to 7^2/2$ and $n \to \infty$ ☺

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$\left(\frac{b_n}{a_n}-1\right)k_n = \left(\frac{b_n - a_n}{a_n}\right)k_n = \left(\frac{-2k_n}{a_n}\right)k_n = -2\left[\frac{k_n^2}{a_n}\right] \to -2\left[\frac{7^2}{2}\right] = -7^2$$

MULTIPLY BY $\frac{7^2}{2}$ $\longrightarrow$ $\frac{2k_n^2}{7^2 a_n} = \left[\frac{2k_n^2}{7^2 n}\right]\left[\frac{n}{a_n}\right] \to 1$

$$\frac{k_n^2}{a_n} \to \frac{7^2}{2}$$

Want: $\left(\frac{b_n}{a_n}-1\right)k_n \to -7^2$ and $k_n \to \infty$

Want: $\left(\frac{n^2}{a_n b_n}-1\right)n \to 7^2/2$ and $n \to \infty$

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$
$$b_n := n - k_n \sim n \to \infty$$

$$\left(\frac{n^2}{a_n b_n} - 1\right) n = \left(\frac{n^2 - a_n b_n}{a_n b_n}\right) n = \left(\frac{k_n^2}{a_n b_n}\right) n = \left[\frac{k_n^2}{a_n}\right]\left[\frac{n}{b_n}\right] \to \left[\frac{7^2}{2}\right][1]$$

$$\frac{n}{b_n} \to 1 \qquad 7^2/2$$

SUBTRACT FROM n^2

$$a_n b_n = (n + k_n)(n - k_n) = n^2 - k_n^2$$

$$\frac{k_n^2}{a_n} \to \frac{7^2}{2}$$

$$n^2 - a_n b_n = n^2 - (n^2 - k_n^2) = k_n^2$$

Want: $\left(\frac{b_n}{a_n} - 1\right) k_n \to -7^2$ and $k_n \to \infty$

Want: $\left(\frac{n^2}{a_n b_n} - 1\right) n \to 7^2/2$ and $n \to \infty$

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n + k_n}\frac{\sqrt{2n}}{2}.$$

$$\frac{k_n}{n} \to 0$$

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IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

Pf:

$$(s_n - 1)t_n \to u$$

$$t_n \to \infty$$

IOU: $(s_n - 1)t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

IOU: $\underbrace{(s_n - 1)}_{\delta_n} t_n \to u$ and $t_n \to \infty$ $\Rightarrow$ $s_n^{t_n} \to e^u$

Pf: $\delta_n := s_n - 1 = \dfrac{(s_n - 1)t_n}{t_n} \to 0$

$(s_n - 1)t_n \to u$, $t_n \to \infty$

$$\frac{[\ln(\underbrace{1 + \delta_n}_{s_n})] - [\ln(1)]}{\delta_n} \to \ln'(1) = \frac{1}{1} = 1$$

$$\frac{\ln s_n}{\delta_n} \to 1$$

MULTIPLY

$$\delta_n t_n \to u$$

$$\left[\frac{\ln s_n}{\delta_n}\right][\delta_n t_n] \to [1][u]$$

EXP

$$\ln[s_n^{t_n}] = [\ln s_n]t_n \to u$$

$$s_n^{t_n} \to e^u$$

QED

STOP