Financial Mathematics

Pricing/hedging in many subperiods

Part 1

Kyle wants right, but not obligation, to buy 5000 shares of ABC for \$5000, 30 days from now.

Gail, seller

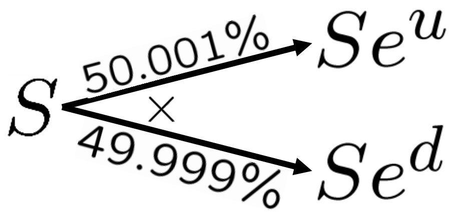
Call option

$N :=$ number of seconds in 30 days

$= 30 \times 24 \times 60 \times 60$

Gail selects:

N-subperiod 50.001-49.999 CRR model

stock price: 

(each second, independently)

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stock price: $S \xrightarrow{50.001\%} Se^u$, $S \xrightarrow{49.999\%} Se^d$ (×)

independent

ln(stock pr): $s \xrightarrow{50.001\%} s+u$, $s \xrightarrow{49.999\%} s+d$ (+)

independent

Market analyst: annual vol = 0.200002881086

annual drift = 0.03399864624

(one second) drift $= 0.03399864624/N_0$

(one second) volatility $= 0.200002881086/\sqrt{N_0}$

$N_0 :=$ number of seconds in one year

$$= 365 \times 24 \times 60 \times 60$$

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stock price: S → (50.001%) Se^u; S → (49.999%) Se^d ($\times$)

change in ln(stock pr): s → (50.001%) $s + u$; s → (49.999%) $s + d$ ($+$)

Market analyst: annual vol = 0.200002881086

annual drift = 0.03399864624

$$(0.50001)u + (0.49999)d = 0.03399864624/N_0$$

$$\text{(one second) volatility} = 0.200002881086/\sqrt{N_0}$$

N_0 := number of seconds in one year $= 365 \times 24 \times 60 \times 60$

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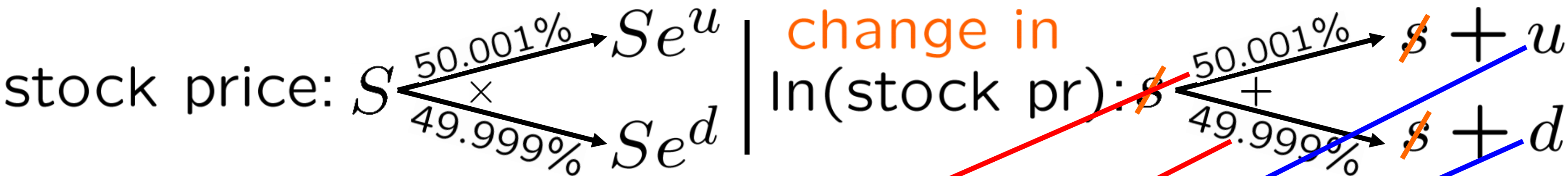

Market analyst: annual vol $= 0.200002881086$

annual drift $= 0.03399864624$

$$(0.50001)u + (0.49999)d = 0.03399864624/N_0$$

2 eq'ns, 2 unknowns

$$\sqrt{(0.50001)(0.49999)}(u - d) = 0.200002881086/\sqrt{N_0}$$

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change in ln(stock pr): s — 50.001% → $s + u$; 49.999% → $s + d$ ($+$)

Calibration:

$$u = 0.00003561536577$$
$$d = -0.00003561463419$$

$$(0.50001)u + (0.49999)d = 0.03399864624/N_0$$

2 eq'ns, 2 unknowns

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stock price: S → 50.001% → Se^u; S → 49.999% → Se^d (×)

$e^u = 1.0000356160000$

$e^d = 0.9999643860000$

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Banker:

(annual) continuous compounding nominal rate $= 0.0315359998802$

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stock price: S → (50.001%) Se^u, (49.999%) Se^d

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$$e^d = 0.9999643860000$$

$N_0 :=$ number of seconds in one year

$$= 365 \times 24 \times 60 \times 60$$

Banker: (or logarithmic risk-free factor)

(annual) continuous compounding nominal rate

$$= 0.0315359998802$$

$$r = 0.0315359998802/N_0$$

$$= 0.0000000009999999624$$

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Gail selects:

N-subperiod 50.001-49.999 CRR model

stock price: $S \xrightarrow{50.001\% \times} Se^u$, $S \xrightarrow{49.999\%} Se^d$

$e^u = 1.0000356160000$

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$N_0 :=$ number of seconds in one year $= 365 \times 24 \times 60 \times 60$

Banker: $(e^u + e^d)/2 = e^r = 1.0000000010000$

(annual) continuous compounding nominal rate

$= 0.0315359998802$

$r = 0.0315359998802/N_0$

$= 0.00000000099999999624$

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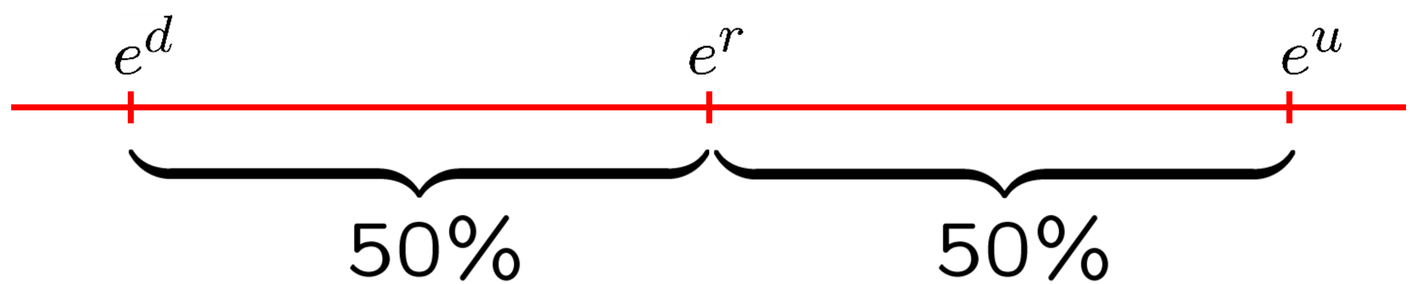

Kyle wants right, but not obligation, to buy 5000 shares of ABC for $5000, 30 days from now.

Gail, seller

Call option

Assume: Initial price = $1/share.

$$e^u = 1.0000356160000$$

$$e^d = 0.9999643860000$$

$$e^r = 1.0000000010000$$

$$e^u = 1.0000356160000$$
$$e^r = 1.0000000010000$$
$$e^d = 0.9999643860000$$

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Gail, seller

Call option

Assume: Initial price = \$1/share.

drift-vol assumption

Each second, price changes
either by a factor of 1.000035616 $= e^u$
or by a factor of 0.999964386. $= e^d$

The one-second risk-free factor is 1.000000001. $= e^r$

$$e^u = 1.0000356160000$$
$$e^r = 1.0000000010000$$
$$e^d = 0.9999643860000$$

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Goal: Find the "right" price, *i.e.*, the price that can be used to set up a "perfect hedge".

Difficulty: $30 \times 24 \times 60 \times 60$ adjustments

Salvation: The Central Limit Theorem!

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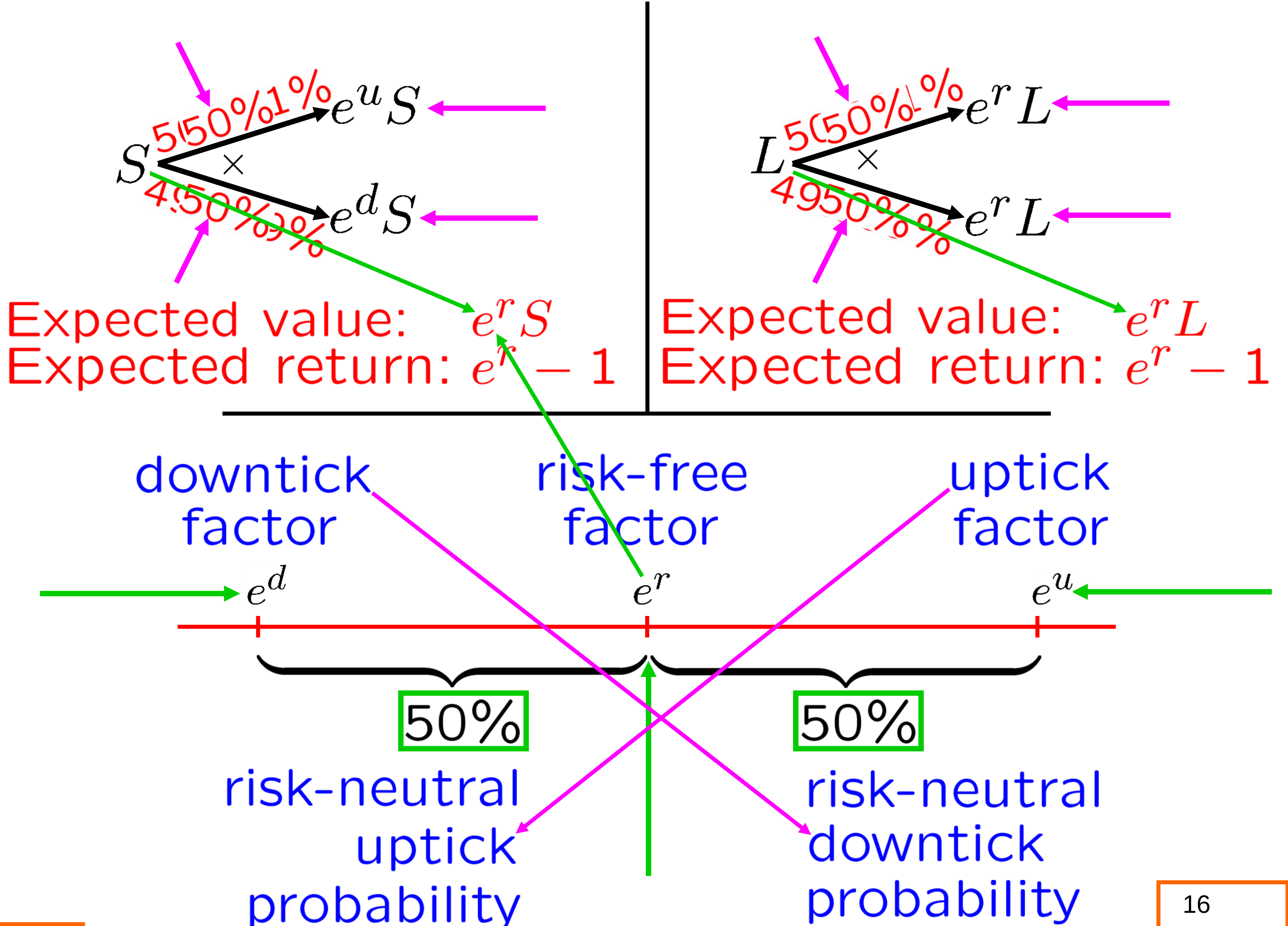
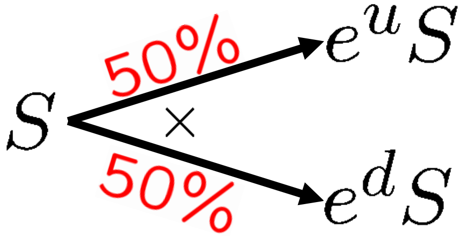
Payoff function:

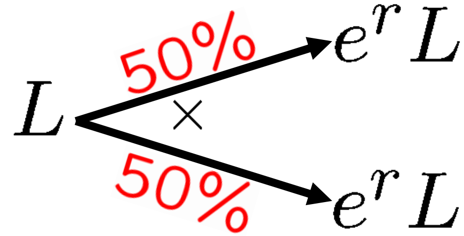
$$f(S) = (5000S - 5000)_+$$

Exercise: Graph f.



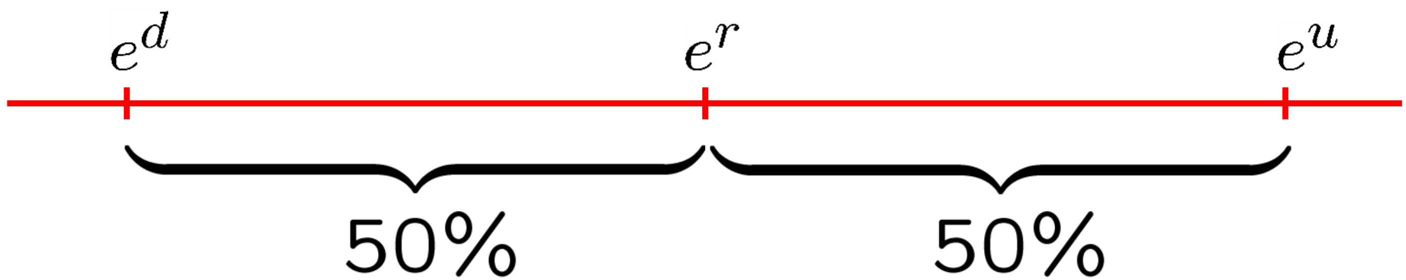
S
50%
$e^u S$
×
50%
$e^d S$
Expected value: $e^r S$
Expected return: $e^r - 1$
L
50%
$e^r L$
×
$e^r L$
Expected value: $e^r L$
Expected return: $e^r - 1$
downtick factor
risk-free factor
uptick factor
e^d
e^r
e^u
50%
50%
risk-neutral uptick probability
risk-neutral downtick probability





Expected value: $e^r S$	Expected value: $e^r L$
Expected return: $e^r - 1$	Expected return: $e^r - 1$

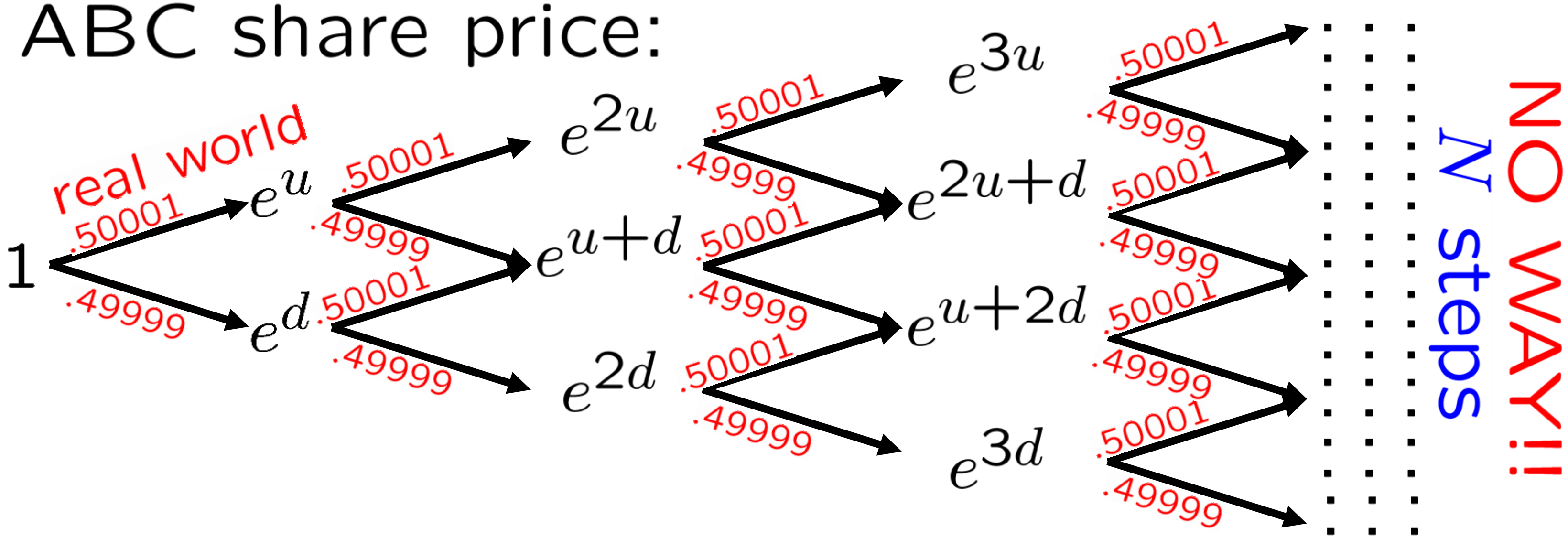
EQUAL



risk-neutral uptick probability

risk-neutral downtick probability

ABC share price:



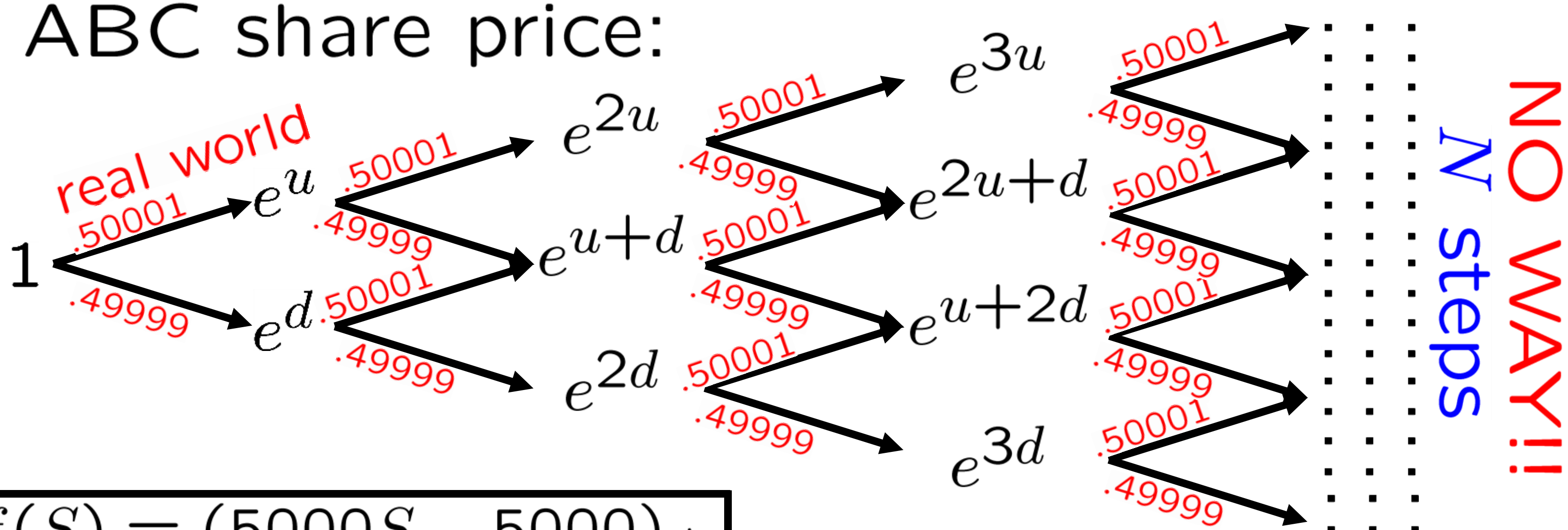
Ending ABC share price:

$$e^{Nu}$$
$$e^{(N-1)u+d}$$
$$e^{(N-2)u+2d}$$
$$\vdots$$
$$e^{Nd}$$

$$N := 30 \times 24 \times 60 \times 60$$
$$= 2,592,000$$

$$1.000035616 =: e^{u}$$
$$0.999964386 =: e^{d}$$

ABC share price:



$f(S) = (5000S - 5000)_+$

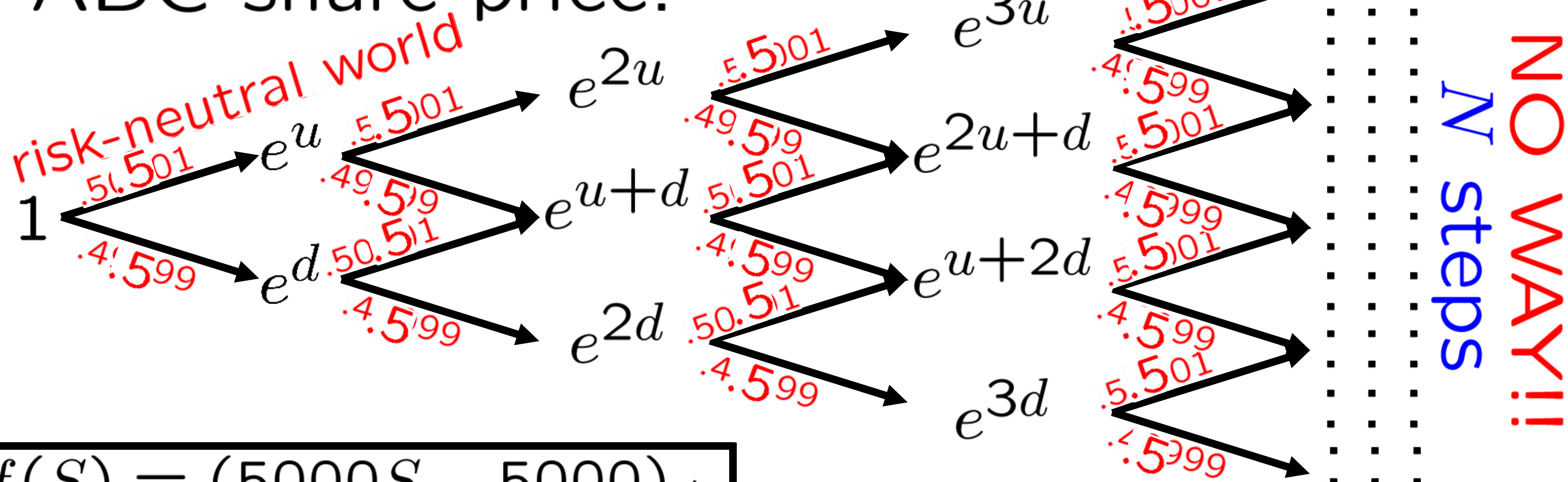
Ending ABC share price:

$$e^{Nu}$$
$$e^{(N-1)u+d}$$
$$e^{(N-2)u+2d}$$
$$\vdots$$
$$e^{Nd}$$

Contingent claim:

$$f(e^{Nu})$$
$$f(e^{(N-1)u+d})$$
$$f(e^{(N-2)u+2d})$$
$$\vdots$$
$$f(e^{Nd})$$

ABC share price:



$f(S) = (5000S - 5000)_+$

$V :=$ price of option

$=$ initial value of hedging portfolio

Contingent claim:

$f(e^{Nu})$

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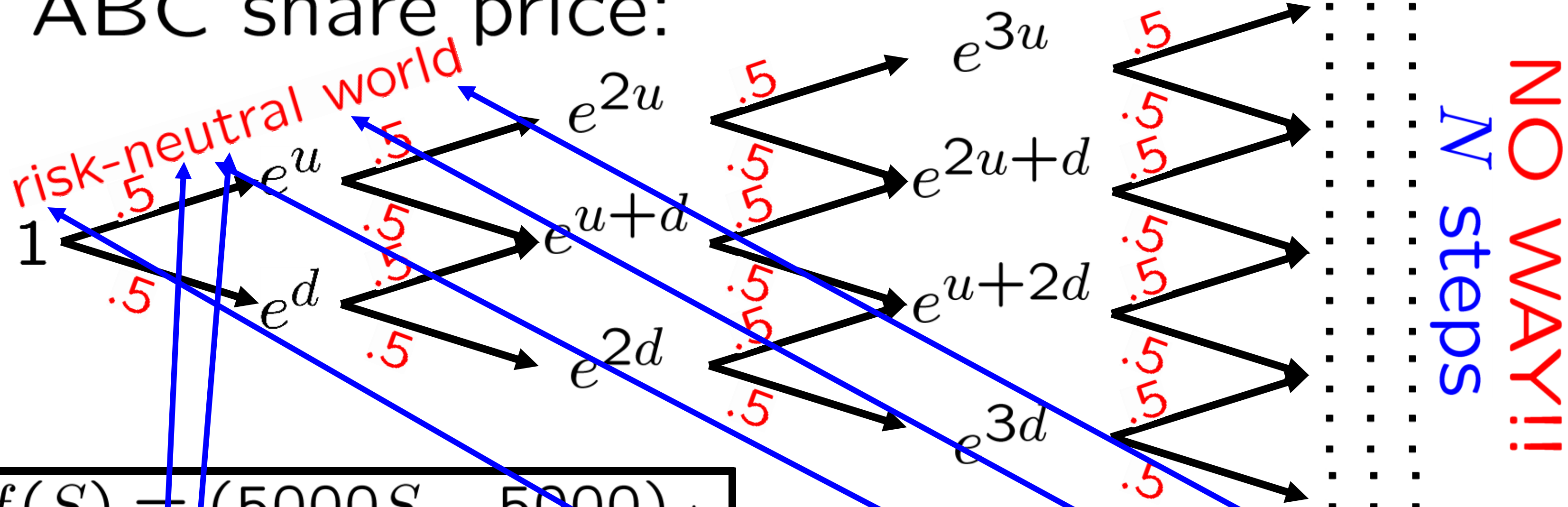
$\vdots$

$f(e^{Nd})$

ABC share price:



$f(S) = (5000S - 5000)_+$

V := price of option
= initial value of hedging portfolio

$e^{rN}V$ =
expected final value of hedging portfolio =
expected contingent claim

Contingent claim:

$f(e^{Nu})$	prob. N, 0
$f(e^{(N-1)u+d})$	prob. $N-1$, 1
$f(e^{(N-2)u+2d})$	prob. $N-2$, 2
⋮	⋮
$f(e^{Nd})$	prob. 0, N

$$\text{prob.}_{j,N-j} = \binom{N}{j} (0.5)^j (0.5)^{N-j}$$

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(e^{Hu+Td})$,
30 days from now.

$e^{rN}V$ = expected payout =: E

$V = e^{-rN}E$ equal

$f(S) = (5000S - 5000)_+$

V := price of option
Goal = initial value of hedging portfolio

$e^{rN}V$ =
expected final value of hedging portfolio =
expected contingent claim

Contingent claim:

$f(e^{Nu})$	prob. N, 0
$f(e^{(N-1)u+d})$	prob. $N-1$, 1
$f(e^{(N-2)u+2d})$	prob. $N-2$, 2
$\vdots$	$\vdots$
$f(e^{Nd})$	prob. 0, N

$$\underset{j,N-j}{\text{prob.}} = \binom{N}{j}(0.5)^j(0.5)^{N-j}$$

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(e^{Hu+Td})$,
30 days from now.

$e^r = 1.000000001$
$N = 2,592,000$

$e^{rN}V =$ expected payout $=: E =$???

Hard problem

expected value problem

$V = e^{-rN}E$
$=$ discounted expected payout

$V :=$ price of option
$=$ initial value of hedging portfolio

$e^{rN}V =$

expected final value of hedging portfolio $=$

expected contingent claim

Contingent claim:

$f(e^{Nu})$ prob. $N, 0$

$f(e^{(N-1)u+d})$ prob. $N-1, 1$

$f(e^{(N-2)u+2d})$ prob. $N-2, 2$

$\vdots$ $\vdots$

$f(e^{Nd})$ prob. $0, N$

$\text{prob.}_{j,N-j} = \binom{N}{j}(0.5)^j(0.5)^{N-j}$

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(e^{Hu+Td})$,
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$$e^r = 1.000000001$$
$$N = 2,592,000$$

$$e^{rN}V = \text{expected payout} =: E = ???$$
$$V = e^{-rN}E$$
$$= \text{discounted expected payout}$$

Easier problem:

probability problems,
then expected value problems

Compute the probability that
$$-\sqrt{N} < H - T < \sqrt{N}.$$