Financial Mathematics

First proof of Black-Scholes

$f(x) := (x - K)_+$

growth of stock $\rightarrow e^X = S_T/S_0$

$S_0 \times$

$S_0 e^X = S_T$

$S_t :=$ price of a stock at time t

call option pays: $f(S_T)$ at time T

logarithmic growth

$X := \ln(S_T/S_0)$

$f(x) := (x - K)_+$

growth of stock $\longrightarrow e^X = S_T/S_0$

$S_0 e^X = S_T$

$S_t :=$ price of a stock at time t

call option pays: $f(S_T)$ at time T

logarithmic growth

$X := \ln(S_T/S_0)$

PAYOUT: $f(S_0 e^X)$

$e^r =$ number of dollars in bank at time T,
if \$1 is invested at time 0

In the simplest models, $X :\rightarrow$ a PCRV

Market analyst gives us: $\mu := \mathrm{E}[\boxed{X}]$

$\sigma := \mathrm{SD}[\boxed{X}]$

Banker gives us: r

Fix $p, q \in (0,1)$ s.t. $p + q = 1$.

In our nth model, $X :\rightarrow X_n \in \sum^n \mathcal{B}_q^p$

n-subperiod (p,q) CRR

PAYOUT: $f(S_0 e^{X_n})$

$f(x) := (x - K)_+$

$S_t :=$ price of a stock at time t

call option pays: $f(S_T)$ at time T

$X := \ln(S_T/S_0)$ PAYOUT: $f(S_0 e^X)$

$e^r =$ number of dollars in bank at time T,

if \$1 is invested at time 0

In the simplest models, $X :\to$ a PCRV

Market analyst gives us: $\mu := \mathrm{E}[\boxed{X_n}]$

$\sigma := \mathrm{SD}[\boxed{X_n}]$

Banker gives us: r

Fix $p, q \in (0,1)$ s.t. $p + q = 1$.

In our nth model, $X :\to X_n \in \sum^n \mathcal{B}_q^p$

n-subperiod (p,q) CRR

PAYOUT: $f(S_0 e^{X_n})$

Market analyst gives us: $\mu := \mathsf{E}[X_n]$

$\sigma := \mathsf{SD}[X_n]$

Banker gives us: r

Fix $p, q \in (0,1)$ s.t. $p + q = 1$.

In our nth model, $X_n \in \sum^n \mathcal{B}_q^p$

Calibration: Find u_n, d_n s.t.

Market analyst gives us: $\mu := \mathsf{E}[X_n]$

$\sigma := \mathsf{SD}[X_n]$

Banker gives us: r

Fix $p, q \in (0,1)$ s.t. $p + q = 1$.

In our nth model, $X_n \in \sum^n \mathcal{B}_q^p$

Market analyst gives us: $\mu := \mathsf{E}[X_n]$

$\sigma := \mathsf{SD}[X_n]$

Banker gives us: r

Fix $p, q \in (0,1)$ s.t. $p + q = 1$.

In our nth model, $X_n \in \sum^n \mathcal{B}_q^p$

more calibration later ...

Calibration: Find u_n, d_n s.t. $\mathsf{E}[\sum^n \mathcal{B}_{q,d_n}^{p,u_n}] = \mu$

$d_n < u_n$

$\mathsf{SD}[\sum^n \mathcal{B}_{q,d_n}^{p,u_n}] = \sigma$

$$\boxed{X_n \in \sum^n \mathcal{B}_{q,d_n}^{p,u_n}}$$

$\mathsf{E}[X_n] = \mu$

$\mathsf{SD}[X_n] = \sigma$

Market analyst gives us: $\mu := \mathsf{E}[X_n]$

$\sigma := \mathsf{SD}[X_n]$

Banker gives us: r

Fix $p, q \in (0, 1)$ s.t. $p + q = 1$.

In our nth model, $X_n \in \sum^n \mathcal{B}_q^p$

Calibration: Find u_n, d_n s.t. $\mathsf{E}[\sum^n \mathcal{B}_{q,d_n}^{p,u_n}] = \mu$

$d_n < u_n$

$\mathsf{SD}[\sum^n \mathcal{B}_{q,d_n}^{p,u_n}] = \sigma$

$$\boxed{X_n \in \sum^n \mathcal{B}_{q,d_n}^{p,u_n}}$$

IOU: $\forall$suff. large n,

$e^{d_n} < e^{r/n} < e^{u_n}$

Risk-neutral world:

$$\underbrace{e^{d_n} \quad\quad e^{r/n}}_{p_n}\underbrace{\quad\quad e^{u_n}}_{q_n}$$

Calibration: Find u_n, d_n s.t. $\mathsf{E}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}} \qquad d_n < u_n \qquad \mathsf{SD}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

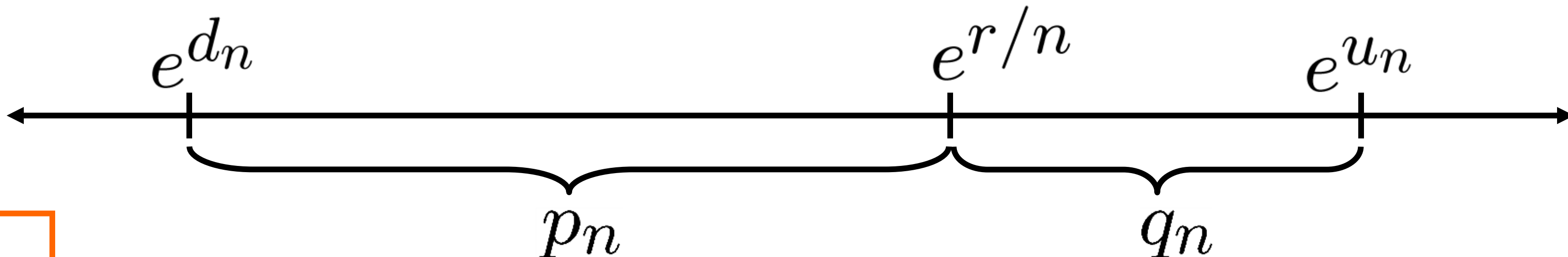
Risk-neutral world:

$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

Calibration: Find u_n, d_n s.t. $\mathsf{E}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}} \qquad d_n < u_n \qquad \mathsf{SD}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

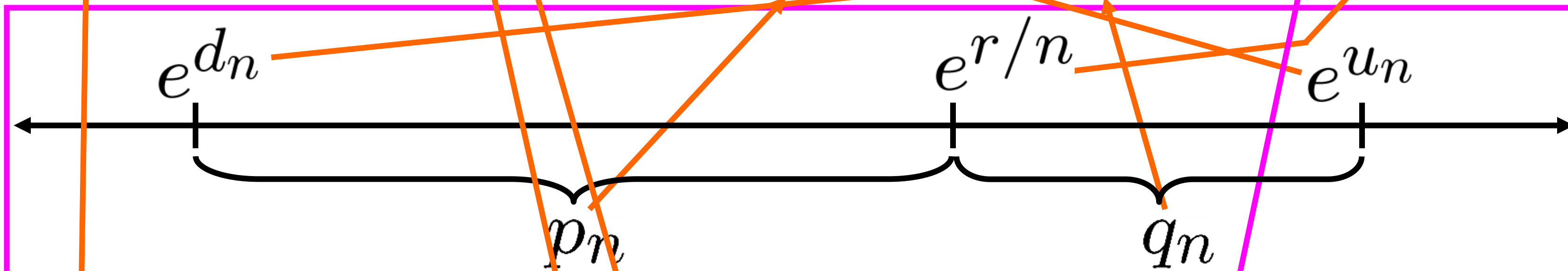
Risk-neutral world:



Calibration: Find u_n, d_n s.t. $\mathrm{E}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}} \quad d_n < u_n \qquad \mathrm{SD}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

Risk-neutral world: $p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$



$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}} \qquad q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

same values
different probabilities

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,\boxed{u_n}}_{q_n,\boxed{d_n}}} \qquad \mathrm{E}[e^{\tilde{X}_n}] = (e^{r/\cancel{n}})^{\cancel{n}} = e^r$$

$$e^{\tilde{X}_n} \in \boxed{\prod^n} \mathcal{B}^{p_n,\boxed{e^{u_n}}}_{q_n,\boxed{e^{d_n}}} \qquad \mathrm{E}[\mathcal{B}^{p_n,e^{u_n}}_{q_n,e^{d_n}}] = e^{r/n}$$

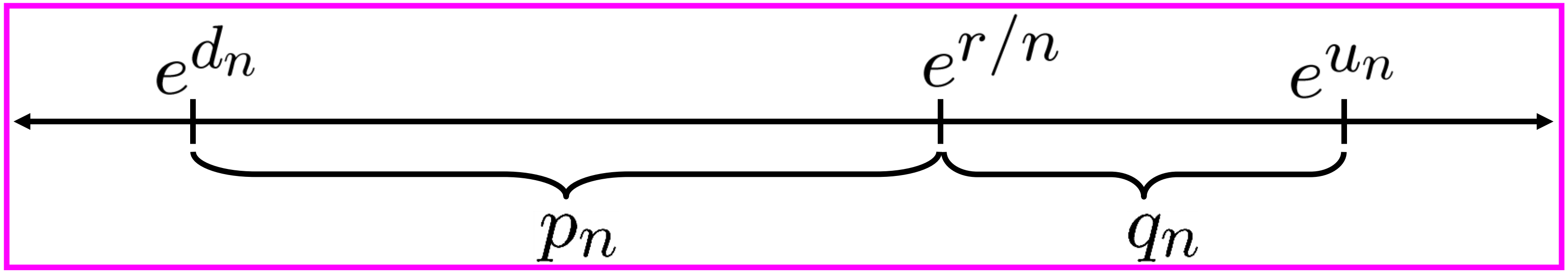
$S_0 e^{X} = S_T$ | Modeled real world: $S_0 e^{X_n} = S_T$ | RN world: $S_0 e^{\tilde{X}_n} = S_T$

Calibration: Find u_n, d_n s.t. $\mathsf{E}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}} \quad d_n < u_n$$

$$\mathsf{SD}[\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

Risk-neutral world:



$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}}$$

The expected stock growth matches the bank.

$$q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

Each dollar grows to e^r dollars, in expectation.

$$\mathsf{E}[e^{\tilde{X}_n}] = (e^{r/\cancel{n}})^{\cancel{n}} = e^r$$

$$e^{\tilde{X}_n} \in \prod^n \mathcal{B}^{p_n,e^{u_n}}_{q_n,e^{d_n}} \qquad \mathsf{E}[\mathcal{B}^{p_n,e^{u_n}}_{q_n,e^{d_n}}] = e^{r/n}$$

$S_0 e^{X} = S_T$ | Modeled real world: $S_0 e^{X_n} = S_T$ | RN world: $S_0 e^{\tilde{X}_n} = S_T$

Modeled real world:

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}}$$

PAYOUT: $f(S_0 e^{X_n})$

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

Risk-neutral world:

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

PAYOUT: $f(S_0 e^{\tilde{X}_n})$

$S_0 e^X = S_T$ | Modeled real world: $S_0 e^{X_n} = S_T$ | RN world: $S_0 e^{\tilde{X}_n} = S_T$

Modeled real world:

$$\boxed{X_n \in \sum^n \mathcal{B}^{p,u_n}_{q,d_n}}$$

PAYOUT: $f(S_0 e^{X_n})$

OPTION PRICE$_n$:
value of hedge at time 0

PAYOUT:
value of hedge at time T

Risk-neutral world:

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

PAYOUT: $f(S_0 e^{\tilde{X}_n})$

OPTION PRICE$_n$:
value of hedge at time 0
$\times [e^r] =$

expected
PAYOUT:
expected
value of hedge at time T

[OPTION PRICE$_n$][e^r]
= E[PAYOUT]
= E[$f(S_0 e^{\tilde{X}_n})$]

equal
same hedging stragtegy

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$[\text{OPTION PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})]$ use $\triangle$CLT2

$[\text{OPTION PRICE}_n][e^r]$

$= \mathsf{E}[f(S_0 e^{\tilde{X}_n})]$

$$\mu_n := \mathsf{E}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{E}[\tilde{X}_n]$$

$$\sigma_n := \mathsf{SD}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{SD}[\tilde{X}_n] \qquad \boxed{\tilde{X}_n \in \textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$$[\mathsf{OPTION\ PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})]$$ use △CLT2

$$n\left(\mathsf{E}[\mathcal{B}^{p,u_n}_{q,d_n}]\right) = n(pu_n + qd_n)$$

$$\mathsf{E}[\textstyle\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$$

continue with calibration. . .

$$\mathsf{SD}[\textstyle\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

$$\sqrt{n}\left(\mathsf{SD}[\mathcal{B}^{p,u_n}_{q,d_n}]\right) = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n := \mathsf{E}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{E}[\tilde{X}_n]$$

$$\sigma_n := \mathsf{SD}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{SD}[\tilde{X}_n] \qquad \boxed{\tilde{X}_n \in \textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$$[\text{OPTION PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})]$$ use $\triangle$CLT2 (later)

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

Calibrate, *i.e.*, solve for u_n, d_n.

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Compute RN drift and vol.

$$n\left(\mathsf{E}[\mathcal{B}^{p,u_n}_{q,d_n}]\right) = n(pu_n + qd_n)$$

$$\mathsf{E}[\textstyle\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \mu$$

$$\mathsf{SD}[\textstyle\sum^n \mathcal{B}^{p,u_n}_{q,d_n}] = \sigma$$

$$\sqrt{n}\left(\mathsf{SD}[\mathcal{B}^{p,u_n}_{q,d_n}]\right) = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n := \mathsf{E}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{E}[\tilde{X}_n]$$

$$\sigma_n := \mathsf{SD}[\textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathsf{SD}[\tilde{X}_n]$$

$$\boxed{\tilde{X}_n \in \textstyle\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$$[\text{OPTION PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})]$$

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\sigma_n \to \sigma$

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

NO μ

Th'm (ΔCLT2):

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

$$\mu_n := \mathrm{E}[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathrm{E}[\tilde{X}_n]$$

$$\sigma_n := \mathrm{SD}[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}] = \mathrm{SD}[\tilde{X}_n]$$

$$\boxed{\frac{\tilde{X}_n - \mu_n}{\sigma_n} \in \mathcal{S}}$$

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}}$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

Th'm (ΔCLT2): $Y_n := \frac{\tilde{X}_n - \mu_n}{\sigma_n}$

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

$$\frac{\tilde{X}_n - \mu_n}{\sigma_n} \in \sum^n \mathcal{B}^{p_n}_{\boxed{q_n}} \qquad q_n = 1 - p_n$$

$$\boxed{\frac{\tilde{X}_n - \mu_n}{\sigma_n} \in \mathcal{S}}$$

Uptick/downtick *VALUES* change,
but uptick/downtick *PROBABILITIES* do not.

$$\boxed{\tilde{X}_n \in \sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}}$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

Th'm (ΔCLT2): $Y_n := \frac{\tilde{X}_n - \mu_n}{\sigma_n}$

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

against contin, exp-bdd

$$\frac{\tilde{X}_n - \mu_n}{\sigma_n} \to Z \text{ in distribution.}$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

Th'm (ΔCLT2):

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

against contin, exp-bdd

$$\frac{\cancel{\sigma_n}}{\sigma} \times \frac{\tilde{X}_n - \mu_n}{\cancel{\sigma_n}} \to Z \text{ in distribution.}$$

against contin, exp-bdd

$$\boxed{\frac{\cancel{\mu_n} - \nu}{\sigma} + \frac{\tilde{X}_n \cancel{- \mu_n}}{\sigma}} \to Z \text{ in distribution.}$$

1

0

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

Th'm (ΔCLT2):

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

against contin, exp-bdd

$Z_n := \boxed{\frac{\tilde{X}_n - \nu}{\sigma}} \to Z$ in distribution.

against contin, exp-bdd

$\boxed{\frac{\cancel{\mu_n} - \nu}{\sigma} + \frac{\tilde{X}_n - \cancel{\mu_n}}{\sigma}} \to Z$ in distribution.

$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})]$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

Th'm (ΔCLT2):

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

Then $Y_n \to Z$ in distribution.

against contin, exp-bdd

$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \to Z$ in distribution.

$\sigma Z_n + \nu = \tilde{X}_n$

$g(Z_n) := f(S_0 e^{\sigma Z_n + \nu})$

$= f(S_0 e^{\tilde{X}_n})$

$g(\underset{Z_n}{x}) := f(S_0 e^{\sigma \overset{Z_n}{x} + \nu})$

[OPTION PRICE$_n$][e^r] $= \mathrm{E}[f(S_0 e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$

$\mu = n(pu_n + qd_n)$

$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$

$\mu_n = n(p_n u_n + q_n d_n)$

$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

Def'n: $Z_n \to Z$ $\underbrace{\text{in distribution}}^{\text{against contin, exp-bdd}}$ means:

for any $\overset{\text{contin, exp-bdd}}{\cancel{\text{contin., bounded}}}$ $\phi : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{E}[\phi(Z_n)] \to \int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx.$$

against contin, exp-bdd

$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \to Z$ in distribution.

$$\sigma Z_n + \nu = \tilde{X}_n \qquad g(Z_n) := f(S_0 e^{\sigma Z_n + \nu})$$

$$g(x) := f(S_0 e^{\sigma x + \nu}) \qquad = f(S_0 e^{\tilde{X}_n})$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$$
$$\sigma_n \to \sigma$$

Def'n: $Z_n \to Z$ $\underbrace{\text{in distribution}}^{\text{against contin, exp-bdd}}$ means:

for any ~~contin., bounded~~ contin, exp-bdd $\phi : \mathbb{R} \to \mathbb{R}$,

g is exp-bdd

$\phi :\to g$ $$\mathrm{E}[\phi(Z_n)] \to \int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx.$$

against contin, exp-bdd

$$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \to Z \text{ in distribution.}$$

$$(S_0 e^{\sigma x + \nu} - K)_+$$

$$\|$$

$$g(x) := f(S_0 e^{\sigma x + \nu}) \longleftarrow f(x) = (x - K)_+$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

against contin, exp-bdd

Def'n: $Z_n \to Z$ $\underbrace{\text{in distribution}}$ means:

for any $\overset{\text{contin, exp-bdd}}{\cancel{\text{contin., bounded}}}$ $\phi : \mathbb{R} \to \mathbb{R}$,

g is exp-bdd

$\phi :\to g$

$$\mathsf{E}[\phi(Z_n)] \to \int_{-\infty}^{\infty} [\phi(x)][h(x)]\,dx.$$

against contin, exp-bdd

$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \to Z$ in distribution. $h(x) = e^{-x^2/2}/\sqrt{2\pi}$

CALCULUS PROBLEM

$$\mathsf{E}[g(Z_n)] \to \int_{-\infty}^{\infty} [g(x)][h(x)]\,dx = \overset{\text{IOU}}{\cdots} = \begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix} e^r$$

version zero

$$g(x) := f(S_0 e^{\sigma x + \nu}) \leftarrow f(x) = (x - K)_+$$

$$[\text{OPTION PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})] = \mathsf{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

$$[\text{OPTION PRICE}_n][\cancel{e^r}] \longrightarrow \underset{\text{version zero}}{\begin{bmatrix}\text{Black-}\\ \text{Scholes}\\ \text{Formula}\end{bmatrix}}\cancel{e^r}$$

Next topic: IOUs

$$\text{OPTION PRICE}_n \longrightarrow \begin{matrix}\text{Black-}\\ \text{Scholes}\\ \text{Formula}\end{matrix}$$

against contin, exp-bdd

$$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \longrightarrow Z \text{ in distribution.}$$

$$\mathrm{E}[g(Z_n)] \longrightarrow \int_{-\infty}^{\infty} [g(x)][h(x)]\,dx \overset{\text{IOU}}{=} \cdots = \underset{\text{version zero}}{\begin{bmatrix}\text{Black-}\\ \text{Scholes}\\ \text{Formula}\end{bmatrix}} e^r$$

$$g(x) := f(S_0 e^{\sigma x + \nu}) \qquad f(x) = (x - K)_+$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$$p_n \longrightarrow p, \qquad q_n \longrightarrow q$$
$$\mu_n \longrightarrow \underbrace{r - (\sigma^2/2)}_{\nu}$$
$$\sigma_n \longrightarrow \sigma$$