Financial Mathematics

IOUs in the proof of Black-Scholes

$[\text{OPTION PRICE}_n][\cancel{e^r}] \longrightarrow \left[\begin{matrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{matrix}\right]_{\substack{\text{version}\\\text{zero}}} \cancel{e^r}$

This topic: IOUs

$\text{OPTION PRICE}_n \longrightarrow$ Black-Scholes Formula

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against contin, exp-bdd

$Z_n := \frac{\tilde{X}_n - \nu}{\sigma} \longrightarrow Z$ in distribution.

$$\mathrm{E}[g(Z_n)] \longrightarrow \int_{-\infty}^{\infty} [g(x)][h(x)]\,dx = \overset{\text{IOU}}{\cdots} = \left[\begin{matrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{matrix}\right]_{\substack{\text{version}\\\text{zero}}} e^r$$

$g(x) := f(S_0 e^{\sigma x + \nu})$ $\qquad f(x) = (x - K)_+$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0 e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \longrightarrow p, \quad q_n \longrightarrow q$

$\mu_n \longrightarrow \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \longrightarrow \sigma$

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$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0 e^{\sigma x+\nu} - K)_+][e^{-x^2/2}]\,dx$$

$$g(x) = (S_0 e^{\sigma x+\nu} - K)_+$$

CALCULUS PROBLEM

$h(x) = e^{-x^2/2}/\sqrt{2\pi}$

$$\mathsf{E}[g(Z_n)] \to \int_{-\infty}^{\infty}[g(x)][h(x)]\,dx = \overset{\text{IOU}}{\cdots} = \begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix} e^r$$

version zero

$$g(x) := f(S_0 e^{\sigma x+\nu}) \qquad f(x) = (x - K)_+$$

$$[\text{OPTION PRICE}_n][e^r] = \mathsf{E}[f(S_0 e^{\tilde{X}_n})] = \mathsf{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$
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$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$$
$$\sigma_n \to \sigma$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0e^{\sigma x+\nu}-K)_+][e^{-x^2/2}]\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\boxed{-d_-}}^{\infty}[(S_0e^{\sigma x+\nu}-K)_{\boxed{+}}][e^{-x^2/2}]\,dx$$

exercise: $[S_0e^{\sigma x+\nu}-K]_{x:\to -d_-}=0$

$$\mathrm{E}[g(Z_n)]\to\int_{-\infty}^{\infty}[g(x)][h(x)]\,dx\overset{\text{IOU}}{=}\cdots=\begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix}e^r$$

$g(x):=f(S_0e^{\sigma x+\nu})$ $\qquad f(x)=(x-K)_+$ version zero

$$[\text{OPTION PRICE}_n][e^r]=\mathrm{E}[f(S_0e^{\tilde{X}_n})]=\mathrm{E}[g(Z_n)]$$

$$\mu=n(pu_n+qd_n)$$
$$\sigma=\sqrt{n}\sqrt{pq}(u_n-d_n)$$

$$\mu_n=n(p_nu_n+q_nd_n)$$
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Girsanov's Theorem and more:

IOU

$p_n\to p,\qquad q_n\to q$

$\mu_n\to\underbrace{r-(\sigma^2/2)}_{\nu}$

$\sigma_n\to\sigma$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0e^{\sigma x+\nu}-K)_+][e^{-x^2/2}]\,dx$$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty}[(S_0e^{\sigma x+\nu}\cancel{-K})\quad][e^{-x^2/2}]\,dx\right)$$

$$-\frac{K}{\sqrt{2\pi}}\int_{-d_-}^{\infty}e^{-x^2/2}\,dx=\boxed{-K[\Phi(d_-)]}$$

$$\mathrm{E}[g(Z_n)]\to\int_{-\infty}^{\infty}[g(x)][h(x)]\,dx\overset{\text{IOU}}{=\cdots=}\begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix}_{\substack{\text{version}\\\text{zero}}}e^r$$

$$g(x):=f(S_0e^{\sigma x+\nu})\qquad f(x)=(x-K)_+$$

$$[\text{OPTION PRICE}_n][e^r]=\mathrm{E}[f(S_0e^{\tilde{X}_n})]=\mathrm{E}[g(Z_n)]$$

$$\mu=n(pu_n+qd_n)$$

$$\sigma=\sqrt{n}\sqrt{pq}(u_n-d_n)$$

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Girsanov's Theorem and more:

IOU

$$p_n\to p,\qquad q_n\to q$$

$$\mu_n\to\underbrace{r-(\sigma^2/2)}_{\nu}$$

$$\sigma_n\to\sigma$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0e^{\sigma x+\nu}-K)_+][e^{-x^2/2}]\,dx$$

$x :\to x+\sigma$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{\boxed{-d_- - \sigma}}^{\infty}[(S_0e^{\cancel{\sigma x}\boxed{\overset{\sigma^2/2}{+\nu}}}\boxed{\cancel{-K}})\;][e^{-x^2/2}]\,dx\right)$$

$= \cancel{(\sigma^2/2)} + [r - \cancel{(\sigma^2/2)}] = r$

$\| \text{exercise}$

$-d_+$

$\boxed{-K[\Phi(d_-)]}$

$$\mathrm{E}[g(Z_n)] \to \int_{-\infty}^{\infty}[g(x)][h(x)]\,dx = \overset{\text{IOU}}{\cdots} = \begin{bmatrix}\text{Black-}\\ \text{Scholes}\\ \text{Formula}\end{bmatrix}e^r$$

version zero

$$g(x) := f(S_0e^{\sigma x+\nu}) \qquad f(x) = (x-K)_+$$

$$[\text{OPTION PRICE}_n][e^r] = \mathrm{E}[f(S_0e^{\tilde{X}_n})] = \mathrm{E}[g(Z_n)]$$

$$\mu = n(pu_n + qd_n)$$

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$$\mu_n = n(p_nu_n + q_nd_n)$$

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Girsanov's Theorem and more:

IOU

$p_n \to p, \quad q_n \to q$

$\mu_n \to \underbrace{r - (\sigma^2/2)}_{\nu}$

$\sigma_n \to \sigma$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0e^{\sigma x+\nu}-K)_+][e^{-x^2/2}]\,dx$$

$=r$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{\boxed{-d_- -\sigma}}^{\infty}[(S_0e^{\cancel{\sigma x}\,\overbrace{\boxed{\sigma^2/2+\nu}}^{=r}}\,\boxed{\cancel{-K}})\;][e^{-x^2/2}]\,dx\right)\;\boxed{-K[\Phi(d_-)]}$$

$$\begin{gathered}\| \\ -d_+\end{gathered}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{-d_+}^{\infty}[S_0e^{r}][e^{-x^2/2}]\,dx\right)-K[\Phi(d_-)]$$

$$\int_{-\infty}^{\infty}[g(x)][h(x)]\,dx \overset{\text{IOU}}{=\!\!=} \cdots = \begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix}e^r$$

$$\int_{-\infty}^{\infty}[g(x)][h(x)]\,dx \overset{\text{IOU}}{=\!\!=} \cdots = \begin{bmatrix}\text{Black-}\\\text{Scholes}\\\text{Formula}\end{bmatrix}e^r$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(S_0 e^{\sigma x+\nu}-K)_+][e^{-x^2/2}]\,dx$$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{\underbrace{\boxed{-d_- -\sigma}}_{\|\atop -d_+}}^{\infty}[(S_0 e^{\cancel{\sigma x}\boxed{\overset{\sigma^2/2}{+\nu}}} \boxed{\cancel{-K}})\quad][e^{-x^2/2}]\,dx\right)$$

$=^{r}$

$$\boxed{-K[\Phi(d_-)]}$$

$$=\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-d_+}^{\infty}[\boxed{S_0 e^{r}}][e^{-x^2/2}]\,dx\right)}_{\boxed{S_0 e^{r}}[\Phi(d_+)]} - \boxed{K}[\Phi(d_-)]$$

$K' e^{r} = K$

FACTOR OUT e^{r}

version zero

$$=\left(\boxed{S_0[\Phi(d_+)] - K'[\Phi(d_-)]}\right)e^{r}$$

$$\int_{-\infty}^{\infty}[g(x)][h(x)]\,dx \overset{\text{IOU}}{=} \cdots = \boxed{\begin{bmatrix}\text{Black-}\\ \text{Scholes}\\ \text{Formula}\end{bmatrix}} e^{r}$$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}}$$

$$q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

exercise

calibration complete

IOU: $\forall$suff. large n,
$e^{d_n} < e^{r/n} < e^{u_n}$

Calibrate,
i.e., solve for u_n, d_n.

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$
$\mu_n \to r - (\sigma^2/2)$
$\sigma_n \to \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$\left(u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n}\right) \times\sqrt{n} \qquad \left(d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}\right) \times\sqrt{n}$$

$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}}$$

$$u_n\sqrt{n} = \sigma\alpha + \underbrace{\boxed{\frac{\mu}{\sqrt{n}}}}_{\to 0} \to \sigma\alpha$$

$$q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

IOU: $\forall$suff. large n,
$e^{d_n} < e^{r/n} < e^{u_n}$

$$d_n\sqrt{n} = -\sigma\beta + \overbrace{\boxed{\frac{\mu}{\sqrt{n}}}}^{\to 0} \to -\sigma\beta$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_nu_n + q_nd_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_nq_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

$\alpha := \sqrt{q/p}$

$\beta := \sqrt{p/q}$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n} \qquad p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}}$$

$$q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

$$u_n\sqrt{n} \to \quad \sigma\alpha \qquad \to \sigma\alpha$$

IOU: $\forall$suff. large n, $e^{d_n} < e^{r/n} < e^{u_n}$

$$(r/n)\sqrt{n} \to$$

$$/\!/$$

$$r/\sqrt{n}$$

$$d_n\sqrt{n} \to -\sigma\beta \qquad \to -\sigma\beta$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \to p$, $q_n \to q$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n = \frac{e^{r/n} - e^{d_n}}{e^{u_n} - e^{d_n}}$$

$$q_n = \frac{e^{u_n} - e^{r/n}}{e^{u_n} - e^{d_n}}$$

$$u_n\sqrt{n} \to \sigma\alpha$$
$$\vee$$
$$(r/n)\sqrt{n} \to 0$$
$$= r/\sqrt{n}$$
$$\vee$$
$$d_n\sqrt{n} \to -\sigma\beta$$

IOU: $\forall$suff. large n,
$$e^{d_n} < e^{r/n} < e^{u_n}$$

$\forall$suff. large n,

NOW EXPONENTIATE: $d_n\cancel{\sqrt{n}} < (r/n)\cancel{\sqrt{n}} < u_n\cancel{\sqrt{n}}$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$
$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$\forall c, k, \exists \omega_n \to 0$ s.t.

$c :\to \sigma\alpha$

$k :\to \mu$

$\omega_n :\to \psi_n$

$$\exp\left(\frac{c}{\sqrt{n}} + \frac{k}{n}\right) = 1 + \frac{c}{\sqrt{n}} + \boxed{\frac{\omega_n}{\sqrt{n}}}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

asymptotics?

$\forall c, k, \ \exists \varepsilon_n \to 0$ s.t.

$$\exp\left(\frac{c}{\sqrt{n}} + \frac{k}{n}\right) = 1 + \left(\frac{c}{\sqrt{n}} + \boxed{\frac{k}{n}}\right) + \varepsilon_n\left(\frac{c}{\sqrt{n}} + \boxed{\frac{k}{n}}\right)$$

$$= 1 + \frac{c}{\sqrt{n}} + \boxed{\boxed{\frac{k/\sqrt{n}}{\sqrt{n}}} + \varepsilon_n\left(\frac{c}{\sqrt{n}} + \boxed{\frac{k/\sqrt{n}}{\sqrt{n}}}\right)}$$

$$// \quad \frac{\omega_n}{\sqrt{n}}$$

$$\omega_n := (k/\sqrt{n}) + \varepsilon_n c + \varepsilon_n(k/\sqrt{n}) \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$
$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$c :\to \sigma\alpha$

$k :\to \mu$

$\forall c, k, \exists \omega_n \to 0$ s.t.

$\omega_n :\to \psi_n$

$$\exp\left(\frac{c}{\sqrt{n}} + \frac{k}{n}\right) = 1 + \frac{c}{\sqrt{n}} + \frac{\omega_n}{\sqrt{n}}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

asymptotics?

$$e^{u_n} = 1 + \frac{\sigma\alpha}{\sqrt{n}} + \frac{\psi_n}{\sqrt{n}}$$

$$\psi_n \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \to p}$, $\boxed{q_n \to q}$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$\forall c, k, \exists \omega_n \to 0$ s.t.

$\omega_n :\to \chi_n$

$c :\to -\sigma\beta$
$k :\to \mu$

$$\exp\left(\frac{c}{\sqrt{n}} + \frac{k}{n}\right) = 1 + \frac{c}{\sqrt{n}} + \frac{\omega_n}{\sqrt{n}}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

asymptotics?

$$e^{u_n} = 1 + \frac{\sigma\alpha}{\sqrt{n}} + \frac{\psi_n}{\sqrt{n}}$$

$$e^{d_n} = 1 - \frac{\sigma\beta}{\sqrt{n}} + \frac{\chi_n}{\sqrt{n}}$$

$\chi_n, \psi_n \to 0$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$\mu_n \longrightarrow r - (\sigma^2/2)$

$\sigma_n \longrightarrow \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$\forall \cancel{c}, k, \exists \omega_n \to 0$ s.t.

$c :\to 0$

$k :\to r$

$$\exp\left(\cancel{\frac{c}{\sqrt{n}}} + \frac{k}{n}\right) = 1 + \cancel{\frac{c}{\sqrt{n}}} + \frac{\omega_n}{\sqrt{n}}$$

$\omega_n :\to \phi_n$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

asymptotics?

$$e^{u_n} = 1 + \frac{\sigma\alpha}{\sqrt{n}} + \frac{\psi_n}{\sqrt{n}}$$

$$e^{r/n} = 1 + \frac{\phi_n}{\sqrt{n}}$$

$$e^{d_n} = 1 - \frac{\sigma\beta}{\sqrt{n}} + \frac{\chi_n}{\sqrt{n}}$$

$\phi_n, \chi_n, \psi_n \to 0$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_nu_n + q_nd_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_nq_n}(u_n - d_n)$$

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$\boxed{p_n \to p}$, $\boxed{q_n \to q}$

$\mu_n \to r - (\sigma^2/2)$

$\sigma_n \to \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$\forall c, k, \exists \omega_n \to 0$ s.t.

$$\exp\left(\frac{c}{\sqrt{n}} + \frac{k}{n}\right) = 1 + \frac{c}{\sqrt{n}} + \frac{\omega_n}{\sqrt{n}}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

asymptotics?

$$p_n \times \left(e^{u_n} = 1 + \frac{\sigma\alpha}{\sqrt{n}} + \frac{\psi_n}{\sqrt{n}}\right)$$

ADD

$$q_n \times \left(e^{d_n} = 1 - \frac{\sigma\beta}{\sqrt{n}} + \frac{\chi_n}{\sqrt{n}}\right)$$

$$e^{r/n} = 1 + \frac{\phi_n}{\sqrt{n}}$$

$$p_n e^{u_n} + q_n e^{d_n} = 1 + \frac{p_n\sigma\alpha - q_n\sigma\beta}{\sqrt{n}} + \frac{\delta_n}{\sqrt{n}}$$

$$\delta_n := p_n\psi_n + q_n\chi_n$$

$$\delta_n, \phi_n, \chi_n, \psi_n \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$
$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$p_n e^{u_n} + q_n e^{d_n} = e^{r/n}$$

$$e^{r/n} = 1 + \frac{\phi_n}{\sqrt{n}}$$

$$e^{r/n} = 1 + \frac{\psi_n}{\sqrt{n}}$$

$$\|$$

$$p_n e^{u_n} + q_n e^{d_n} = 1 + \frac{p_n\sigma\alpha - q_n\sigma\beta}{\sqrt{n}} + \frac{\delta_n}{\sqrt{n}}$$

$$\delta_n, \phi_n, \chi_n, \psi_n \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
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Girsanov's Theorem and more:

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$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$
$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$\boxed{\frac{p_n\sigma\alpha - q_n\sigma\beta}{\sqrt{n}} + \frac{\delta_n}{\sqrt{n}}} = \boxed{\frac{\psi_n}{\sqrt{n}}}$$

$$e^{r/n} = 1 + \boxed{\frac{\psi_n}{\sqrt{n}}}$$
$$\|$$
$$p_n e^{u_n} + q_n e^{d_n} = 1 + \boxed{\frac{p_n\sigma\alpha - q_n\sigma\beta}{\sqrt{n}} + \frac{\delta_n}{\sqrt{n}}}$$

$\delta_n, \phi_n, \chi_n, \psi_n \to 0$

$$\mu = n(pu_n + qd_n)$$
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IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$\mu_n \longrightarrow r - (\sigma^2/2)$

$\sigma_n \longrightarrow \sigma$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$\sqrt{n} \times \left(\frac{p_n\sigma\alpha - q_n\sigma\beta}{\sqrt{n}} + \frac{\delta_n}{\sqrt{n}} = \frac{\psi_n}{\sqrt{n}} \right)$$

$$p_n\sigma\alpha - q_n\sigma\beta \boxed{+\,\delta_n} = \psi_n$$

$$(p_n\alpha - q_n\beta)\boxed{\sigma} = p_n\boxed{\sigma}\alpha - q_n\boxed{\sigma}\beta = \psi_n - \delta_n$$

$$p_n\alpha - q_n\beta = \frac{\psi_n - \delta_n}{\boxed{\sigma}} \to 0$$

$$\delta_n, \phi_n, \chi_n, \psi_n \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_nu_n + q_nd_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_nq_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$
$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p} \qquad p\alpha = \sqrt{p^2}\sqrt{q/p}$$

$$\beta := \sqrt{p/q} \qquad q\beta = \sqrt{q^2}\sqrt{p/q}$$

$$\beta := \sqrt{p/q}$$

$$p_n\alpha - q_n\beta \to 0$$

$$p_n\alpha - q_n\beta \qquad \to 0$$

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$

$$\sigma_n \longrightarrow \sigma$$

$$\alpha := \sqrt{q/p} \qquad p\alpha = \sqrt{p^2}\sqrt{q/p} = \sqrt{pq}$$

SUBTRACT

$$\beta := \sqrt{p/q} \qquad q\beta = \sqrt{q^2}\sqrt{p/q} = \sqrt{pq}$$

$$p\alpha - q\beta = 0$$

SUBTRACT

$$p_n\alpha - q_n\beta \to 0$$

$$p\alpha - q\beta = 0$$

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_nu_n + q_nd_n)$$

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Girsanov's Theorem and more:

IOU

$\boxed{p_n \longrightarrow p}$, $\boxed{q_n \longrightarrow q}$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$

$$\sigma_n \longrightarrow \sigma$$

$\alpha := \sqrt{q/p}$

$\beta := \sqrt{p/q}$

SUBTRACT

$$
\begin{aligned}
0 &= 1 - 1 \\
s_n &:= p_n - p \\
\hline
-s_n &= q_n - q
\end{aligned}
$$

SUBTRACT

$$
\begin{aligned}
p_n\alpha - q_n\beta &\to 0 \\
p\alpha - q\beta &= 0 \\
\hline
s_n(\alpha+\beta) = s_n\alpha + (+s_n)\beta &\to 0
\end{aligned}
$$

DIVIDE BY $\alpha + \beta$

$p_n - p = s_n \to 0$ ADD p $p_n \to p$

MULTIPLY BY -1

$q_n - q = -s_n \to 0$ ADD q $q_n \to q$

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem

☺ and more: ☺

IOU

$\boxed{p_n \to p}$, $\boxed{q_n \to q}$

$\mu_n \to r - (\sigma^2/2)$

$\boxed{\sigma_n \to \sigma}$

$$\alpha := \sqrt{q/p}$$

$$\beta := \sqrt{p/q}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \cancel{\frac{\mu}{n}} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \cancel{\frac{\mu}{n}}$$

$$u_n - d_n = \frac{\sigma(\alpha+\beta)}{\boxed{\sqrt{n}}}$$

$$(u_n - d_n)\sqrt{n} = \sigma(\alpha+\beta)$$

exercise

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more:

IOU

$p_n \longrightarrow p,\quad q_n \longrightarrow q$

$\mu_n \longrightarrow r - (\sigma^2/2)$

$\boxed{\sigma_n \longrightarrow \sigma}$

$$\alpha := \sqrt{q/p}$$
$$\beta := \sqrt{p/q}$$

$$u_n = \frac{\sigma\alpha}{\sqrt{n}} + \frac{\mu}{n} \qquad d_n = \frac{-\sigma\beta}{\sqrt{n}} + \frac{\mu}{n} \qquad u_n - d_n = \frac{\sigma(\alpha+\beta)}{\sqrt{n}}$$

$$(u_n - d_n)\sqrt{n} = \sigma(\alpha+\beta)$$

$$\sigma_n \to \sigma$$

ADD σ

$$\sigma_n - \sigma = \boxed{\sqrt{n}}(\sqrt{p_n q_n} - \sqrt{pq})(u_n - d_n)$$
$$= \underbrace{(\sqrt{p_n q_n} - \sqrt{pq})}_{\downarrow\, 0}\underbrace{(u_n - d_n)\sqrt{n}}_{=\,\sigma(\alpha+\beta)} \to 0$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more: ☺ ☺

IOU

$p_n \to p$, $q_n \to q$

$\boxed{\mu_n \to r - (\sigma^2/2)}$

$\boxed{\sigma_n \to \sigma}$ ☺

MULTIPLY

$$\frac{\sigma_n}{\sigma} \to 1$$

$$\frac{\tilde{X}_n - \mu_n}{\sigma_n} \to Z$$ in dist, against contin, exp-bdd

$$\frac{\tilde{X}_n - \mu_n}{\sigma} \to Z$$ in dist, against contin, exp-bdd

$$\mathrm{E}[e^{\tilde{X}_n}] = e^r$$ MULTIPLY BY $e^{-\mu_n}$

Each dollar grows to e^r dollars, in expectation.

The expected stock growth matches the bank.

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem ☺ and more: ☺

IOU

$p_n \to p$, $q_n \to q$

$$\boxed{\mu_n \to r - (\sigma^2/2)}$$

$\sigma_n \to \sigma$ ☺

$\frac{\tilde{X}_n - \mu_n}{\sigma_n} \to Z$ in dist, against contin, exp-bdd

$W_n := \frac{\tilde{X}_n - \mu_n}{\sigma} \to Z$ in dist, against contin, exp-bdd

$\mathrm{E}[e^{\tilde{X}_n}] = e^r$ MULTIPLY BY $e^{-\mu_n}$

$\mathrm{E}[e^{\tilde{X}_n - \mu_n}] = e^{r - \mu_n}$

$\|$

$\mathrm{E}[e^{\sigma W_n}] \xrightarrow{\text{CLT}} \mathrm{E}[e^{\sigma \boxed{Z}}]$ Z not yet def'd, so...

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem ☺ and more: ☺ IOU

$p_n \to p$, $q_n \to q$

$\boxed{\mu_n \to r - (\sigma^2/2)}$

$\sigma_n \to \sigma$ ☺

$$\frac{\tilde{X}_n-\mu_n}{\sigma_n} \longrightarrow Z$$ in dist, against contin, exp-bdd

$$W_n := \frac{\tilde{X}_n-\mu_n}{\sigma} \longrightarrow Z$$ in dist, against contin, exp-bdd

$$\mathrm{E}[e^{\tilde{X}_n}] = e^r$$

"Change every Z to x and then integrate against $h(x)\,dx$, from $-\infty$ to ∞.

$$\mathrm{E}[e^{\tilde{X}_n-\mu_n}] = e^{r-\mu_n}$$

$$h(x) := e^{-x^2/2}/\sqrt{2\pi}$$

$$\mathrm{E}[e^{\sigma W_n}] \xrightarrow{\text{CLT}} \mathrm{E}[e^{\sigma \boxed{Z}}]$$

Z not yet def'd, so...

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sigma x}\, e^{-x^2/2}\, dx$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more: ☺ ☺

IOU

$p_n \longrightarrow p$, $q_n \longrightarrow q$

$$\mu_n \longrightarrow r - (\sigma^2/2)$$

$\sigma_n \longrightarrow \sigma$ ☺

$$\frac{\tilde{X}_n-\mu_n}{\sigma_n} \longrightarrow Z$$ in dist, against contin, exp-bdd

$$W_n := \frac{\tilde{X}_n-\mu_n}{\sigma} \longrightarrow Z$$ in dist, against contin, exp-bdd

$$\mathsf{E}[e^{\tilde{X}_n}] = e^r$$

$$\mathsf{E}[e^{\tilde{X}_n-\mu_n}] = e^{r-\mu_n}$$

$$\|$$

$$\mathsf{E}[e^{\sigma W_n}] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma x}\, e^{-x^2/2}\, dx$$

$$x :\longrightarrow x+\sigma$$

$$\frac{}{\sqrt{2\pi}} \int_{-\infty} e^{\sigma x}\, e^{-x^2/2}\, dx$$

$$\mu = n(pu_n + qd_n)$$
$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$
$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem

☺ and more: ☺

IOU

$p_n \longrightarrow p$, $q_n \longrightarrow q$

$$\boxed{\mu_n \longrightarrow r - (\sigma^2/2)}$$

$\sigma_n \longrightarrow \sigma$ ☺

$$\frac{\tilde{X}_n - \mu_n}{\sigma_n} \longrightarrow Z \text{ in dist, against contin, exp-bdd}$$

$$W_n := \frac{\tilde{X}_n - \mu_n}{\sigma} \longrightarrow Z \text{ in dist, against contin, exp-bdd}$$

$$\mathsf{E}[e^{\tilde{X}_n}] = e^r$$

STOP

$$\mathsf{E}[e^{\tilde{X}_n - \mu_n}] = e^{r - \mu_n}$$

TAKE ln

$$\| $$

$$\mathsf{E}[e^{\sigma W_n}] \xrightarrow{\text{CLT}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\cancel{\sigma x}} e^{-x^2/2}\, dx = e^{\sigma^2/2}$$

$\sigma^2/2$

$x :\to x + \sigma$

$$r - (r - \mu_n) \longrightarrow \underbrace{(\sigma^2/2)}_{r-} \qquad \mu_n \longrightarrow r - (\sigma^2/2)$$

$$\mu = n(pu_n + qd_n)$$

$$\sigma = \sqrt{n}\sqrt{pq}(u_n - d_n)$$

$$\mu_n = n(p_n u_n + q_n d_n)$$

$$\sigma_n = \sqrt{n}\sqrt{p_n q_n}(u_n - d_n)$$

Girsanov's Theorem and more: ☺ ☺

IOU

$p_n \longrightarrow p, \quad q_n \longrightarrow q$

$$\boxed{\mu_n \longrightarrow r - (\sigma^2/2)}$$

$\sigma_n \longrightarrow \sigma$ ☺ ☺