FM 5011 Fall 2009, Midterm #2
Handout date: Thursday 19 November 2009

PRINT NAME:

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 0012, 3 pts.) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **integrable** or L^1 if ...

b. (Topic 0014, 3 pts.) Two Borel spaces $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are **isomorphic** if ...

c. (Topic 0015, 3 pts.) Let μ be a measure on $\mathbb{R}$. The **cumulative distribution function** of μ is the function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \cdots$.

d. (Topic 0016, 3 pts.) Let $[a, b]$ be a compact interval. A function $f : [a, b] \to \mathbb{R}$ is said to be of **bounded variation** if ...

e. (Topic 0017, 3 pts.) If X and Y are two L^2 random variables on a probability space $(\Omega, \mathcal{A}, \mu)$, then the **covariance** of X and Y is defined by $\mathrm{Cov}[X, Y] = \cdots$. (You may use Var in your formula, without defining it.)

II. True or False. (No partial credit.)

a. (Topic 0015, 3 pts.) If f is a continuous function on a compact interval $[a, b]$, then the Riemann and Lebesgue integrals of f (w.r.t. Lebesgue measure) on $[a, b]$ are equal.

b. (Topic 0017, 3 pts.) Any integrable random variable is square integrable.

c. (Topic 0018, 3 pts.) Let X be any random variable on a probability space $(M, \mathcal{B}, \mu)$ and let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{B}$. Let Y represent $E[X|\mathcal{A}]$. Then Y is $\mathcal{A}$-measurable.

d. (Topic 0012, 3 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and let μ be a measure on $\mathbb{R}$. Let C be a closed subset of $\mathbb{R}$ and assume that f is supported on C. Then $f\mu$ is supported on C (*i.e.*, concentrated on C).

e. (Topic 0015, 3 pts.) If two probability measures on $\mathbb{R}$ have the same Fourier transform, then they are equal.

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PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1).

III(2).

III(3).

III(4).

III(5a,b).

III(5c,d).

III(6).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 0013, 10 pts.) Let λ denote Lebesgue measure on $\mathbb{R}$. Find an example of a sequence of integrable functions $f_1, f_2, f_3, \ldots : \mathbb{R} \to \mathbb{R}$ such that BOTH:

- for all $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = 0$; AND
- for all integers $n \geq 1$, $\displaystyle\int_{-\infty}^{\infty} f_n \, d\lambda = 1$.

2. (Topic 0013, 10 pts.) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$. Let λ denote Lebesgue measure on $\mathbb{R}$ and let $\mu := h\lambda$. Compute $\displaystyle\int_{-\infty}^{\infty} \quad x^6 + 2x^4 + 5x^3 \quad d\mu(x)$.

3. (Topic 0017, 10 pts.) Let Z be a standard normal random variable. Find the variance of e^{2Z+3}.

4. (Topic 0018, 10 pts.) Let $\Omega := [0,1] \times [0,1]$ with its standard σ-algebra $\mathcal{B}$. Let λ_1 be Lebesgue measure on $[0,1]$ and let $\mu := \lambda_1 \times \lambda_1$ be the product measure on Ω, so $(\Omega, \mathcal{B}, \mu)$ is a probability space. Define random variables U, V on Ω by $U(x,y) = x$ and $V(x,y) = 3x^5 + 4y^6$. Find a random variable W on Ω such that W represents $\mathrm{E}[V|U]$.

5. Let δ_2 denote the point mass at 2.

a. (Topic 0015, 5 pts.) Let $I := (1, 2)$ be the open interval from 1 to 2. Compute $\delta_2(I)$.

b. (Topic 0015, 5 pts.) Let $J := [1, 2]$ be the closed interval from 1 to 2. Compute $\delta_2(J)$.

5. (continued) Recall that δ_2 is the point mass at 2.

c. (Topic 0015, 5 pts.) Compute $\displaystyle\int_{-\infty}^{\infty} \quad x^3 - 2x^2 + 5 \quad d\delta_2(x)$.

d. (Topic 0015, 5 pts.) Find the Fourier transform of δ_2. (Following the definition used in this class, your answer should be an expression of t.)

6. Let Z be a standard normal random variable.

a. (Topic 0015, 5 pts.) Find the CDF of the distribution of e^{2Z}. Your final answer may involve Φ (the CDF of Z).

b. (Topic 0015, 5 pts.) Find a PDF of the distribution of e^{2Z}. Your final answer should *not* involve Φ or Φ'.