CALCULUS
Polynomials and rational functions

Collecting like terms

We do not typically write

$$\underline{4x^2} + \underline{3} + \underline{6x} + \underline{5} + \underline{2x} + \underline{x^2} \underline{- 10} + \underline{8} \underline{- x}$$

Instead:

$$(4+1)x^2 \quad + \quad (6+2-1)x \quad + \quad (3+5-10+8)$$

$$= 5x^2 + 7x + 6 \quad \text{(decreasing degree)}$$

$$= 6 + 7x + 5x^2 \quad \text{(increasing degree)}$$

SKILL: Collecting like terms

Spp

Collecting like terms

Problem: Expand and collect terms in

$$(3 + t + 4t^2)(5t^7 - 1) + (t + 4)(t + t^8)$$

$$(20 + 1)t^9 + (5 + 4)t^8 + (15)t^7 +$$

$$(\qquad)t^6 + (\qquad)t^5 + (\qquad)t^4 +$$

$$+(\qquad)t^3 + (-4 + 1)t^2 + (-1 + 4)t + (-3)$$

Count terms: $3 \cdot 2 + 2 \cdot 2 = 10$

Collecting like terms

Problem: Expand and collect terms in

$$(3+t+4t^2)(5t^7-1)+(t+4)(t+t^8)$$

$$= \quad (20+1)t^9+(5+4)t^8+(15)t^7+$$

$$(-4+1)t^2+(-1+4)t+(-3)$$

$$(-4+1)t^2+(-1+4)t+(-3)$$

$$= \; 21t^9+9t^8+15t^7-3t^2+3t-3$$

Collecting like terms

Problem: Expand and collect terms in

$$(3+t+4t^2)(5t^7-1)+(t+4)(t+t^8)$$

$$= (20+1)t^9+(5+4)t^8+(15)t^7+ (-4+1)t^2+(-1+4)t+(-3)$$

$$= 21t^9+9t^8+15t^7-3t^2+3t-3 \blacksquare$$

SKILL: Expand and collect terms

Note: Answer is a linear combination of t^9, t^8, t^7, t^6, t^5, t^4, t^3, t^2, t, 1.

Polynomials and rational functions

Def: A **polynomial in** t
is a finite linear comb.
of $1,\ t,\ t^2,\ \ldots$

$$21t^9 + 9t^8 + 15t^7 - 3t^2 + 3t - 3 \blacksquare$$

SKILL: Expand and collect terms

Note: Answer is a linear combination of
$t^9,\ t^8,\ t^7,\ t^6,\ t^5,\ t^4,\ t^3,\ t^2,\ t,\ 1.$

Polynomials and rational functions

SKILL
simplify poly

$$(3+t+4t^2)(5t^7-1) + (t+4)(t+t^8)$$

$$= (20+1)t^9 + (5+4)t^8 + (15)t^7 + (-4+1)t^2 + (-1+4)t + (-3)$$

Polynomial in t? YES
Why or why not?

SKILL
recognize poly

not def'd at $q=-1$

$$\frac{(q+1)(q+2)}{q+1}$$

Polynomial in q? NO
Why or why not?

Domain of a polynomial $= \mathbb{R}$

Def: A **polynomial in** t is a finite linear comb. of 1, t, t^2, $\ldots$

e.g.: $5+3t-2t^2$

$4+t^{1,000,000}$

$3+5t^2-19t^5$

Defs: **polynomial in** x
polynomial in u
polynomial in q
etc. **polynomial**

e.g.: $5+3\bullet-2\bullet^2$

$4+\bullet^{1,000,000}$

$3+5\bullet^2-19\bullet^5$

Polynomials and rational functions

SKILL
simplify poly

$(3+t+4t^2)(5t^7-1)$
$+(t+4)(t+t^8)$

$= (20+1)t^9 + (5+4)t^8 + (15)t^7 + (-4+1)t^2 + (-1+4)t + (-3)$

Polynomial in t? YES

Def: A **polynomial in** t is a finite linear comb. of 1, t, t^2, . . .

Defs: **polynomial in** x
polynomial in u
polynomial in q
etc. **polynomial**

SKILL
recognize poly

not def'd at $q=-1$

$$\frac{(q+1)(q+2)}{q+1}$$

Polynomial in q? NO

Def: A **rational fn** is a quotient of two polys.

polynomial in q
etc. **polynomial**

Polynomials and rational functions

SKILL
simplify poly

$$(20+1)t^9+(5+4)t^8+(15)t^7+ (-4+1)t^2+(-1+4)t+(-3)$$

//

$$(3+t+4t^2)(5t^7-1) +(t+4)(t+t^8)$$

Polynomial in t? YES

Def: A **polynomial in** t is a finite linear comb. of 1, t, t^2, . . .

Defs: **polynomial in** x
polynomial in u
polynomial in q
etc. **polynomial**

SKILL
recognize poly

not def'd at $q=-1$

$$\frac{(q+1)(q+2)}{q+1}$$

Polynomial in q? NO
Rational in q? YES

Def: A **rational fn** is a quotient of two polys.

Defs: **rat'l expr. of** t
rat'l expr. of v
rat'l expr. of b
rat'l expr. of x
etc.

Polynomials and rational functions

Def: A **polynomial in** t is a finite linear comb. of 1, t, t^2, . . .

Defs: **polynomial in** x
polynomial in u
polynomial in q
etc. **polynomial**

Def: A **rational fn** is a quotient of two polys.

Defs: **rat'l expr. of** t
rat'l expr. of v
rat'l expr. of b
rat'l expr. of x
etc.

e.g.: $\dfrac{5+3x-2x^2}{4+x^{1,000,000}}$

$\dfrac{7+2t^9+4t^{10}}{(1-t)(4+t)}$ domain? $t \in \mathbb{R}\backslash\{1,-4\}$

$$\left[\frac{4}{w}-\frac{2w^2-7w}{w^7+4}\right] \Big/ \left[\frac{7}{w^3}-\frac{(1+w)(2w^2-7w)}{w^7}\right]$$

§1.3 Rational expr of w? Why or why not?

Polynomials and rational functions

$$\frac{4w^{14} - 2w^{10} + 7w^9 + 16w^7}{w(w^7+4)(7w^4 - 2w^3 + 5w^2 + 7w)} = \cdots$$

SKILL
simplify rat'l fn

expand and collect terms

$$\| $$

$$\left[\frac{4w^7 - 2w^3 + 7w^2 + 16}{w(w^7+4)}\right]\left[\frac{w^7}{7w^4 - 2w^3 + 5w^2 + 7w}\right]$$

$$\| $$

collect terms — collect terms

$$\left[\frac{(4w^7 + 16) - (2w^3 - 7w^2)}{w(w^7+4)}\right] \Big/ \left[\frac{(7w^4) - (2w^3 + (-7+2)w^2 - 7w)}{w^7}\right]$$

$$\| $$

expand — expand — expand

$$\left[\frac{4(w^7+4)}{w(w^7+4)} - \frac{w(2w^2 - 7w)}{w(w^7+4)}\right] \Big/ \left[\frac{7w^4}{w^7} - \frac{(1+w)(2w^2 - 7w)}{w^7}\right]$$

common denominator — common denominator

$$/\!/$$

$$\left[\frac{4}{w} - \frac{2w^2 - 7w}{w^7+4}\right] \Big/ \left[\frac{7}{w^3} - \frac{(1+w)(2w^2-7w)}{w^7}\right]$$

SKILL
recognize rat'l fn

§1.3 Rational expr of w? YES Why or why not?

Polynomials and rational functions

Let $P(x)$ be a polynomial in x.

SKILL
Find degree of poly

degree of $P(x)$:=
highest power of x appearing in $P(x)$

e.g.: $3x + 4x^5 - 2x + 7$ has degree 5

Constant means degree 0
Constant polynomials: 2, 7, -8, 0, π, *etc.*

Linear means degree 1
Linear polynomials: $2x + 5$, $ex - \sqrt{2}$, πx, *etc.*

Quadratic means degree 2
Quadratic polynomials: $-7x^2 - 4x + 8$, *etc.*

Cubic means degree 3
Cubic polynomials: $2x^3 - \pi x^2 + 6x + 1$, *etc.*

Quartic means degree 4
Quartic polynomials: $8x^4 - 4x^3 + 2x^2 + 4x + 6$, *etc.*

Quintic means degree 5
Quintic polynomials: $4x^5 - \pi x^4 + 2x^3 - ex^2 + 5x - 8$, *etc.*

Polynomials and rational functions

A degree six (sextic) polynomial:

$$9x^6 - 8x^5 + 7x^4 - 6x^3 + 5x^2 - 4x + 3$$

- $9x^6$: **Degree six term**
- $-8x^5$: **Quintic term**
- $+7x^4$: **Quartic term**
- $-6x^3$: **Cubic term**
- $+5x^2$: **Quadratic term**
- $-4x$: **Linear term**
- $+3$: **Constant term**

SKILL
Identify terms of poly

The **coefficients** are the numbers. . .

Polynomials and rational functions

A degree six (sextic) polynomial:

$$9x^6 - 8x^5 + 7x^4 - 6x^3 + 5x^2 - 4x + 3$$

Constant coefficient

Linear coefficient

Quadratic coefficient

Cubic coefficient

Quartic coefficient

Quintic coefficient

Degree six coefficient

SKILL
Identify coefficients of poly

Leading coefficient := the coefficient on the highest degree term.

The **coefficients** are the numbers. . .

Polynomials and rational functions

A degree six (sextic) polynomial:

$\boxed{9}x^6 - 8x^5 + 7x^4 - 6x^3 + 5x^2 - 4x + 3$

SKILL
Identify leading coefficient of poly

Leading coefficient := the coefficient on the highest degree term.

A polynomial is **monic** if its leading coeff. is 1,

§1.3 *e.g.*, $x^2 + 5x - 4$, $x^3 - 4x^2 + 7x - 2$, *etc.*

Polynomials and rational functions

A degree six (sextic) polynomial:

$\boxed{9x^6} - 8x^5 + 7x^4 - 6x^3 + 5x^2 - 4x + 3$

SKILL
Identify leading term of poly

Leading term := the highest degree term

A polynomial is **monic** if its leading coeff. is 1,

§1.3 *e.g.*, $x^2 + 5x - 4$, $x^3 - 4x^2 + 7x - 2$, *etc.*

Heierarchy of functions and expressions

transcendental = non-algebraic

e.g., $\sin x$

algebraic (closed under $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$)

e.g., $\sqrt[4]{\sqrt[5]{x^3 - x} + \sqrt[7]{\dfrac{x^5 - 2x}{3x + 8}}}$

rational except for a single $\sqrt{\ }$,

e.g., $\sqrt{x^2 + 1} - x$

rational (closed under $+$, $-$, $\times$, $\div$)

e.g., $\dfrac{x^2 - 4x + 3}{2x + 7} - \dfrac{1}{x^2 + 1}$

polynomial (closed under $+$, $-$, $\times$)

e.g., $2x^2 - 4x + 5$

constant, linear, quadratic, *etc.*

Heierarchy of functions and expressions

transcendental = non-algebraic

algebraic = evaluable by $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$

algebraic = evaluable by $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$

rational = evaluable by $+$, $-$, $\times$, $\div$

polynomial = evaluable by $+$, $-$, $\times$

poly $\Rightarrow$ rat'l $\Rightarrow$ algebraic $\not\Rightarrow$ transcendental

rational = evaluable by $+$, $-$, $\times$, $\div$

polynomial = evaluable by $+$, $-$, $\times$

Heierarchy of functions and expressions

transcendental = non-algebraic

algebraic = evaluable by $+$, $-$, $\times$, $\div$, $\sqrt[n]{}$

rational = evaluable by $+$, $-$, $\times$, $\div$

polynomial = evaluable by $+$, $-$, $\times$

poly $\Rightarrow$ rat'l $\Rightarrow$ algebraic $\not\Rightarrow$ transcendental

algebraic is the opposite of transcendental

Next: division & synthetic division

Polynomial division

Problem: Divide $2x^5 + 4x^4 - 28x^3 + 4x^2 + 2x - 1$ by $x^3 - 4x^2 + 5x - 2$.

MULTIPLY $\quad 2x^2 + 12x + 10$

$$
\begin{array}{r|l}
x^3 - 4x^2 + 5x - 2 & 2x^5 + 4x^4 - 28x^3 + 4x^2 + 2x - 1 \\
\text{SUBTRACT} & 2x^5 - 8x^4 + 10x^3 - 4x^2 \\
\hline
 & 12x^4 - 38x^3 + 8x^2 + 2x \\
\text{SUBTRACT} & 12x^4 - 48x^3 + 60x^2 - 24x \\
\hline
 & 10x^3 - 52x^2 + 26x - 1 \\
\text{SUBTRACT} & 10x^3 - 40x^2 + 50x - 20 \\
\hline
 & -12x^2 - 24x + 19
\end{array}
$$

Polynomial division

Problem: Divide $2x^5 + 4x^4 - 28x^3 + 4x^2 + 2x - 1$ by $x^3 - 4x^2 + 5x - 2$.

$$
\begin{array}{r|l}
 & 2x^2 + 12x + 10 \\
\hline
x^3 - 4x^2 + 5x - 2 & 2x^5 + 4x^4 - 28x^3 + 4x^2 + 2x - 1 \\
 & \underline{2x^5 - 8x^4 + 10x^3 - 4x^2} \\
 & 12x^4 - 38x^3 + 8x^2 + 2x \\
 & \underline{12x^4 - 48x^3 + 60x^2 - 24x} \\
 & 10x^3 - 52x^2 + 26x - 1 \\
 & \underline{10x^3 - 40x^2 + 50x - 20} \\
 & -12x^2 - 24x + 19
\end{array}
$$

SKILL
polynomial division

$$2x^5 + 4x^4 - 28x^3 + 4x^2 + 2x - 1 = (x^3 - 4x^2 + 5x - 2)(2x^2 + 12x + 10) - 12x^2 - 24x + 19$$

Polynomial division

Problem: Divide $3x^3 - x^2 - 6x + 1$ by $x - 2$.

$$
\begin{array}{r|l}
 & 3x^2 + 5x + 4 \\
\hline
x - 2 & 3x^3 - x^2 - 6x + 1 \\
 & \underline{3x^3 - 6x^2} \\
 & \quad 5x^2 - 6x \\
 & \quad \underline{5x^2 - 10x} \\
 & \qquad 4x + 1 \\
 & \qquad \underline{4x - 8} \\
 & \qquad\quad 9
\end{array}
$$

NEGATE

COPY

Synthetic division:

MULTIPLY

$$
\begin{array}{c|cccc}
 & 3 & 5 & 4 & \\
\hline
2 & 3 & -1 & -6 & 1 \\
 & & 6 & 10 & 8 \\
 & & & & \overline{9}
\end{array}
$$

COPY

ADD ADD ADD

ALTERNATE FORMATTING

Polynomial division

Problem: Divide $3x^3 - x^2 - 6x + 1$ by $x - 2$.

$$
\begin{array}{r|l}
 & 3x^2 + 5x + 4 \\
\hline
x - 2 & 3x^3 - x^2 - 6x + 1 \\
 & \underline{3x^3 - 6x^2} \\
 & 5x^2 - 6x \\
 & \underline{5x^2 - 10x} \\
 & 4x + 1 \\
 & \underline{4x - 8} \\
 & 9
\end{array}
$$

Synthetic division:

$$
\begin{array}{r|rrrr}
 & 3 & 5 & 4 & \\
\hline
2 & 3 & -1 & -6 & 1 \\
 & & 6 & 10 & 8 \\
\hline
 & 3 & 5 & 4 & \boxed{9}
\end{array}
$$

ALTERNATE FORMATTING

Polynomial division

Problem: Divide $3x^3 - x^2 - 6x + 1$ by $x - 2$.

$$
\begin{array}{r|l}
 & 3x^2 + 5x + 4 \\
\hline
x-2 & 3x^3 - x^2 - 6x + 1 \\
 & \underline{3x^3 - 6x^2} \\
 & \quad 5x^2 - 6x \\
 & \quad \underline{5x^2 - 10x} \\
 & \qquad 4x + 1 \\
 & \qquad \underline{4x - 8} \\
 & \qquad\quad 9
\end{array}
$$

Synthetic division:

SKILL
synthetic division of
$x - a$ into polynomial

2	3	−1	−6	1
		6	10	8
	3	5	4	9

$$[3x^3 - x^2 - 6x + 1]_{x:\to 2} = [(x-2)(3x^2 + 5x + 4) + 9]_{x:\to 2}$$
$$= 0 + 9$$
$$= 9$$

Dividing $x-2$ into $p(x)$, remainder is $[p(x)]_{x:\to 2} = p(2)$.
Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

Spp

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

i.e., iff a is a root of p (or zero)

Exercise: Factor $x - 4$ out of

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$

as many times as possible.

4						
	1	−10	21	68	−272	192
		4	−24	−12	224	−192
	1	−6	−3	56	−48	0

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$
$$= (x - 4)(x^4 - 6x^3 - 3x^2 + 56x - 48)$$

Polynomial division

Dividing $x-a$ into $p(x)$, remainder is $p(a)$.

$x-a$ divides evenly into $p(x)$ iff $p(a)=0$

i.e., iff a is a root of p (or zero)

Exercise: Factor $x-4$ out of

$$x^5-10x^4+21x^3+68x^2-272x+192$$

as many times as possible.

4	1	−10	21	68	−272	192
		4	−24	−12	224	−192
4	1	−6	−3	56	−48	0
		4	−8	−44	48	
	1	−2	−11	12	0	

$$x^5-10x^4+21x^3+68x^2-272x+192$$
$$=(x-4)(x^4-6x^3-3x^2+56x-48)$$
$$=(x-4)^2(x^3-2x^2-11x+12)$$

Spp

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

i.e., iff a is a root of p $\underbrace{}_{\text{(or zero)}}$

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4	1	−6	−3	56	−48	0
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4	1	−2	−11	12	0	
		4	8	−12		
	1	2	−3	0		

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$
$$= (x-4)^3(x^2 + 2x - 3)$$

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

i.e., iff a is a root of p (or zero)

Exercise: Factor $x - 4$ out of

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$

as many times as possible.

4	1	−10	21	68	−272	192
		4	−24	−12	224	−192
4	1	−6	−3	56	−48	0
		4	−8	−44	48	
4	1	−2	−11	12	0	
		4	8	−12		
4	1	2	−3	0		
		4	24			
	1	6	21 $\neq 0$			

SKILL
Repeated factoring of $x - a$ from poly

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192 = (x-4)^3(x^2 + 2x - 3)$$ ■

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

i.e., iff a is a root of p (or zero)

Exercise: Factor $x - 4$ out of

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$

as many times as possible.

$$\left[x^2 + 2x - 3\right]_{x:\to 4} = 21 \neq 0$$

$$\begin{array}{r|rrr} 4 & 1 & 2 & -3 \\ & & 4 & 24 \\ \hline & 1 & 6 & 21 \neq 0 \end{array}$$

SKILL
Repeated factoring of $x - a$ from poly

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$
$$= (x-4)^3(x^2 + 2x - 3)$$

Spp

Polynomial division

Dividing $x - a$ into $p(x)$, remainder is $p(a)$.

$x - a$ divides evenly into $p(x)$ iff $p(a) = 0$

i.e., iff a is a root of p (or zero)

Exercise: Factor $x - 4$ out of

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$

as many times as possible.

Note: $x = 4$ is a root of

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$

of **multiplicity** 3.

SKILL
Find the multiplicity
of a root of a poly

SKILL
Repeated factoring
of $x - a$ from poly

$$x^5 - 10x^4 + 21x^3 + 68x^2 - 272x + 192$$
$$= (x - 4)^3(x^2 + 2x - 3)$$ ■

Spp

SKILL

Find domain

Whitman problems

§1.3, p. 13, #1-12

SKILL

Find domain of composite

Whitman problems

§1.3, p. 13, #13

SKILL

words to fn & find domain

Whitman problems

§1.3, p. 13, #14-15