CALCULUS
Miscellaneous precalculus

EXAMPLE: Find where the function

$$g(x) = 12x^3 + 12x^2 - 24x$$

is positive and where it is negative.

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factoring a quadratic . . .

$$\begin{aligned} g(x) &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + 1x - 2) \\ &= 12x(x + 2)(x - 1) \end{aligned}$$

$-2 = (-2) \times (+1)$

$(-2) + (+1) = -1$

$-2 = (+2) \times (-1)$

$(+2) + (-1) = 1$

g is negative on $(-\infty, -2)$,
positive on $(-2, 0)$,
negative on $(0, 1)$
and positive on $(1, \infty)$. ■

SKILL
factoring a polynomial

SKILL
intervals of pos&neg for a factored polynomial

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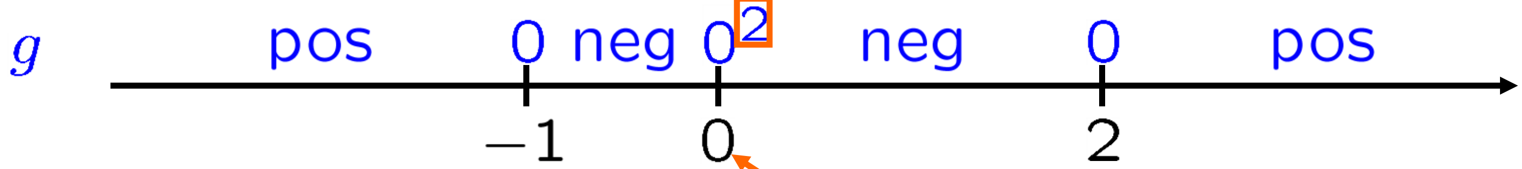
ANOTHER EXAMPLE: Find where the function

$$g(x) = 60x^4 - 60x^3 - 120x^2$$

is positive and where it is negative.

$$\begin{aligned} g(x) &= 60x^4 - 60x^3 - 120x^2 \\ &= 60x^2(x^2 - x - 2) \\ &= 60x^{2}(x-2)(x+1) \end{aligned}$$



this is a root of multiplicity 2

g is positive on $(-\infty, -1)$,
negative on $(-1, 0)$,
negative on $(0, 2)$
and positive on $(2, \infty)$. ■

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factoring a polynomial

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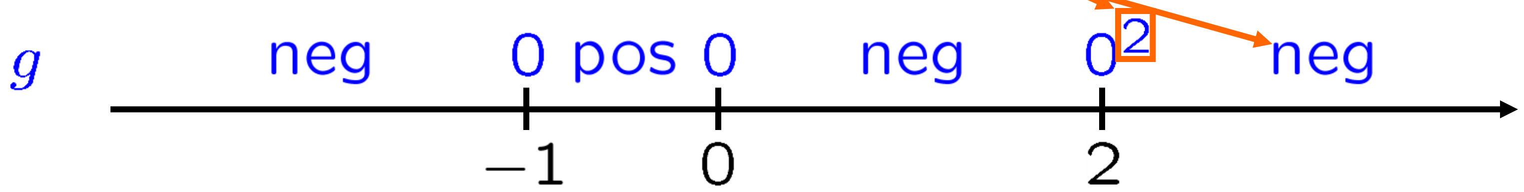
intervals of pos&neg for a factored polynomial

ANOTHER EXAMPLE: Find where the function

$$g(x) = -20x^4 + 60x^3 - 80x$$

is positive and where it is negative.

$$\begin{aligned} g(x) &= -20x^4 + 60x^3 - 80x \\ &= -20x(x^3 - 3x^2 + 4) \\ &= -20x(x-2)^2(x+1) \end{aligned}$$



g is negative on $(-\infty, -1)$,
positive on $(-1, 0)$,
negative on $(0, 2)$
and negative on $(2, \infty)$. ■

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factoring a polynomial

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intervals of pos&neg for a factored polynomial

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ANOTHER EXAMPLE: Find where the function

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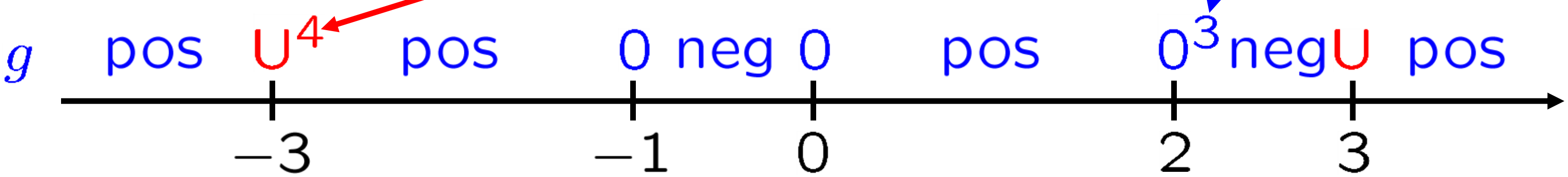
intervals of
pos&neg
for a factored
rational function

$$g(x) = \frac{x(x-2)^3(x+1)}{(x-3)(x+3)^4}$$

is positive and where it is negative.

"U" means "Undefined"



g is positive on $(-\infty, -3)$,
positive on $(-3, -1)$,
negative on $(-1, 0)$,
positive on $(0, 2)$,
negative on $(2, 3)$
and positive on $(3, \infty)$. ■

Next: The binomial theorem . . .

The Binomial Theorem, or. . .

how to expand

$$(x+y)^0, (x+y)^1, (x+y)^2, (x+y)^3, \ldots$$

$$(x+y)^0 \overset{x+y\neq 0}{=} 1$$

$2^0 = 1$ terms

$$(x+y)^1 = x + y$$

$2^1 = 2$ terms

$$(x+y)^2 = x(x + y) + y(x + y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

duplication

$2^2 = 4$ terms

$$(x+y)^3 = x(xx + xy + yx + yy) + y(xx + xy + yx + yy)$$
$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

duplications

$2^3 = 8$ terms

$$(x+y)^4 = x(xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy) + y(xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy) = \text{etc.}$$

duplications

$2^4 = 16$ terms

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Lots of duplications. . . *e.g.*

$$
\begin{aligned}
(x+y)^5 = {} & xxxxx + xxxxy + xxxyx + xxxyy \\
& + xxyxx + xxyxy + xxyyx + xxyyy \\
& + xyxxx + xyxxy + xyxyx + xyxyy \\
& + xyyxx + xyyxy + xyyyx + xyyyy \\
& + yxxxx + yxxxy + yxxyx + yxxyy \\
& + yxyxx + yxyxy + yxyyx + yxyyy \\
& + yyxxx + yyxxy + yyxyx + yyxyy \\
& + yyyxx + yyyxy + yyyyx + yyyyy
\end{aligned}
$$

x^4y

Start over, avoiding duplications. . .

$2^5 = 32$ terms

$$(x+y)^0 \overset{x+y\neq 0}{=} 1 = 1$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy+1y^2$$

$$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$$

$$\begin{aligned}
(x+y)^3 &= (x+y)\quad(x+y)^2 \\
&= (x+y)(1x^2+2xy+1y^2) \\
&= x\,(1x^2+2xy+1y^2) \\
&\quad + y\,(1x^2+2xy+1y^2) \\
&= 1x^3+2x^2y+1xy^2 \\
&\quad + 1x^2y+2xy^2+1y^3 \\
&= 1x^3+3x^2y+3xy^2+1y^3
\end{aligned}$$

$$(x+y)^0 = 1 = 1 \qquad x+y \neq 0$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^3 = 1x^3 + 2x^2y + 1xy^2$$

$$+ \quad 1x^2y + 2xy^2 + 1y^3$$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$= 1x^3 + 2x^2y + 1xy^2$$

$$\begin{array}{ccccc} & & 1 & 2 & 1 \\ \hline + & 1 & 3x^2y & +3xy^2 & +1y^3 \end{array}$$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

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$$(x+y)^0 = 1 = 1 \quad {\scriptstyle x+y\neq 0}$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy+1y^2$$

$$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$$

$$(x+y)^4$$

1	2	1		
	1	2	1	
1	3	3	1	

1	3	3	1	
	1	3	3	1

1	2	1	
	1	2	1
1	3	3	1

$$(x+y)^0 = 1 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4$$

1	2	1	
	1	2	1
1	3	3	1

1	3	3	1	
	1	3	3	1
1	4	6	4	1

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$(x+y)^0 =$ 1 $\quad x+y \neq 0$

$(x+y)^1 = 1x + 1y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

1	2	1		
	1	2	1	
1	3	3	1	

1	3	3	1	
	1	3	3	1
1	4	6	4	1

$(x+y)^0 = 1$ ($x+y \neq 0$)

$(x+y)^1 = 1x + 1y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Start with four xs.

Change an x to a y.

Continue...

until...

four ys.

1	3	3	1	
	1	3	3	1
1	4	6	4	1

Easier: 1 + 3 + 3 + 1

1 4 6 4 1

$(x+y)^0 = 1 \quad x+y \neq 0$

$(x+y)^1 = 1x + 1y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

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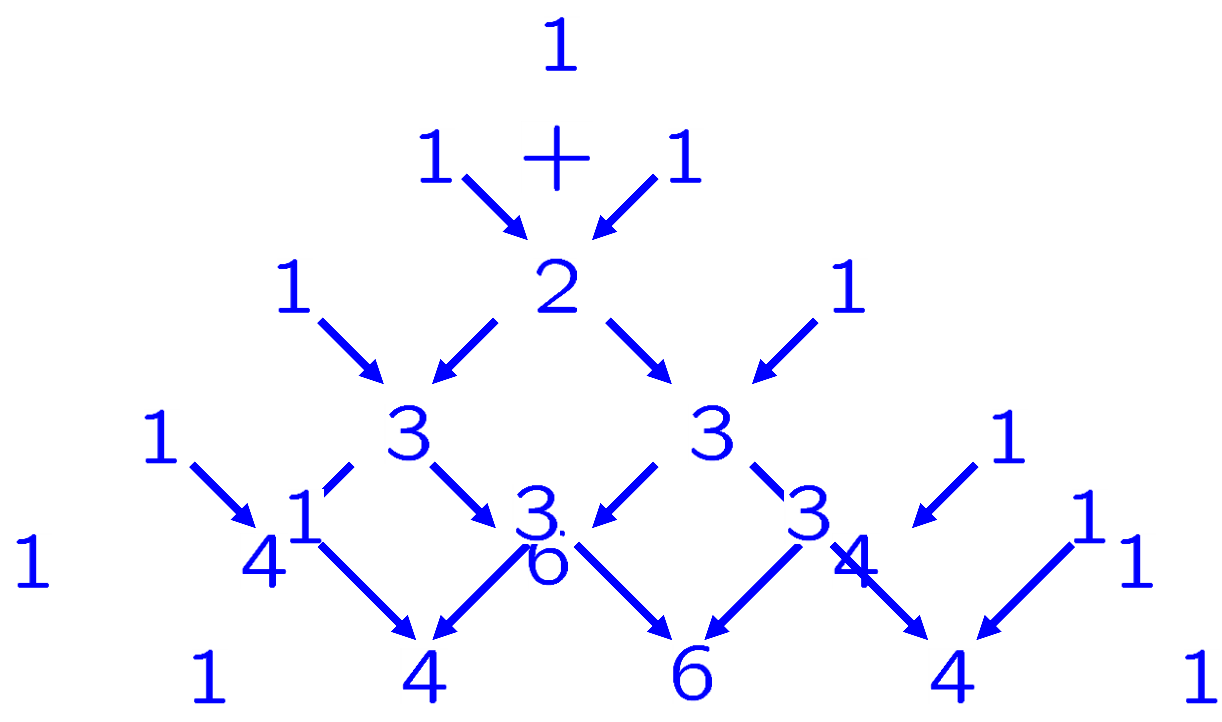

$$(x+y)^0 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = 1x + 1y$$

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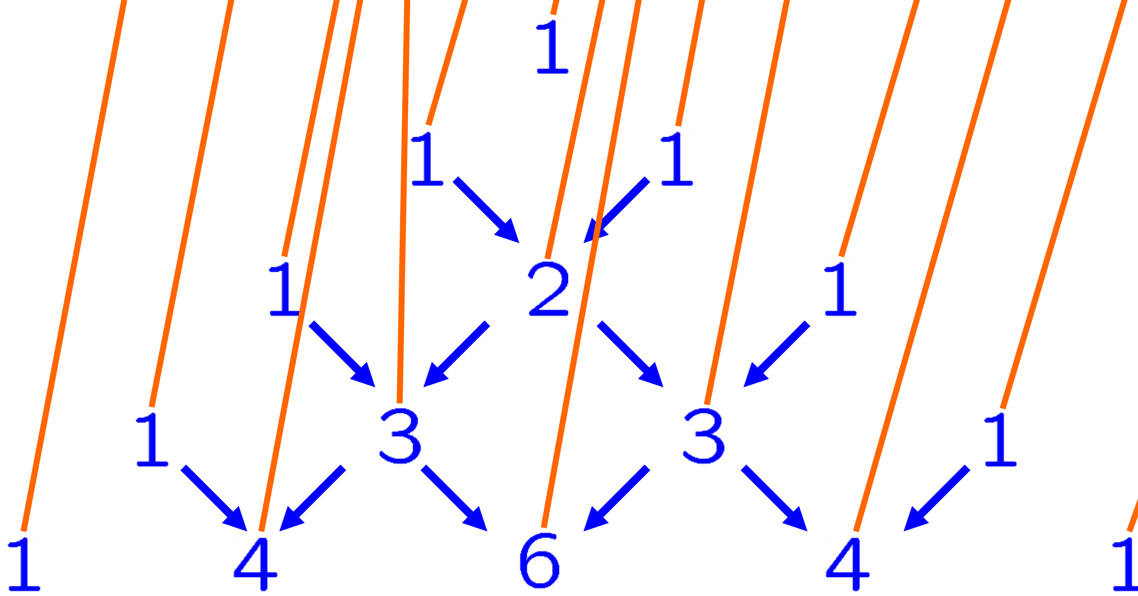

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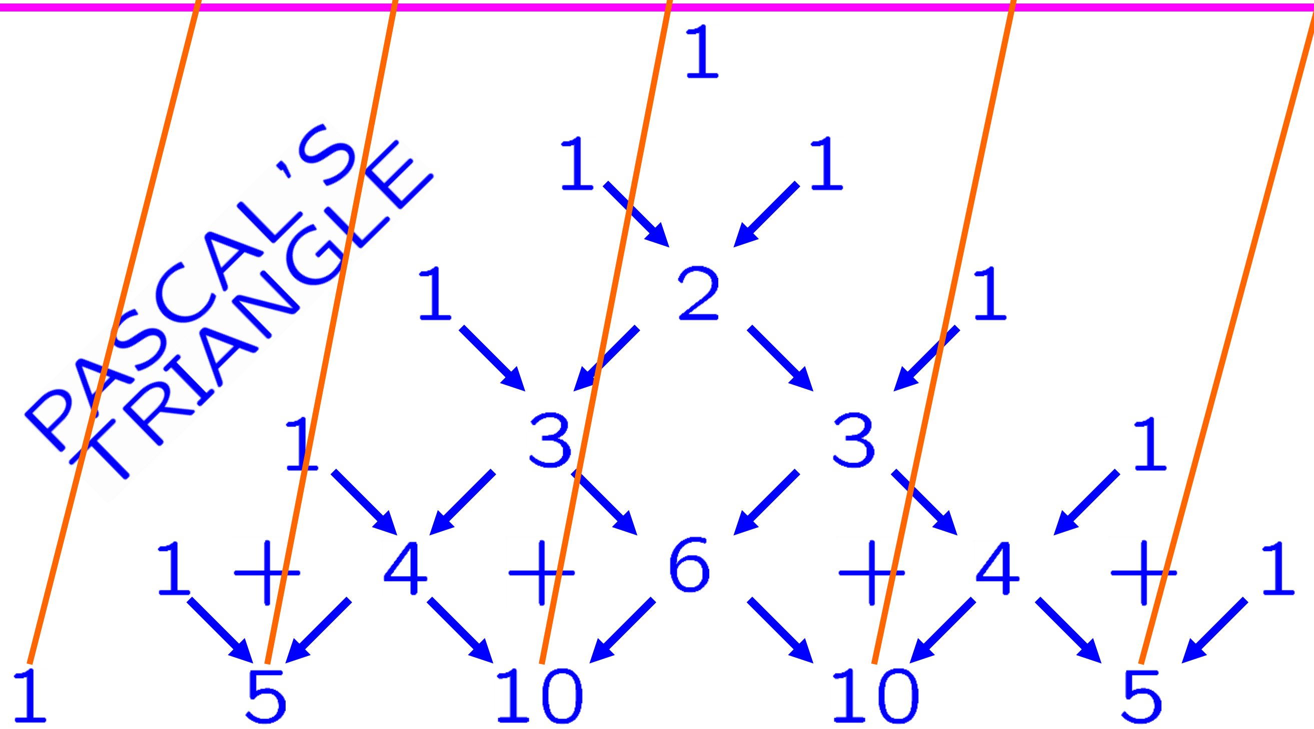
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$



$$(x+y)^0 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

PASCAL'S TRIANGLE

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Next: Rationalizing numerators and denominators . . .

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Rationalize numerator and denominator

SKILL rationalize num&den

Problem: Rationalize the denominator in $\dfrac{1}{\boxed{\sqrt{2}}}$.

irrational

Solution: $\dfrac{1}{\sqrt{2}} = \left[\dfrac{1}{\sqrt{2}}\right]\left[\dfrac{\sqrt{2}}{\sqrt{2}}\right] = \dfrac{\boxed{\sqrt{2}}}{\boxed{2}}$ ■

irrational

rational

Problem: Rationalize the denominator in $\dfrac{4}{\sqrt{7}}$.

Solution: $\dfrac{4}{\sqrt{7}} = \left[\dfrac{4}{\sqrt{7}}\right]\left[\dfrac{\sqrt{7}}{\sqrt{7}}\right] = \dfrac{4\sqrt{7}}{7}$ ■

Rationalize numerator and denominator

SKILL rationalize num&den

Problem: Rationalize the numerator in $\frac{\sqrt{3}}{\sqrt{7}}$.

Solution: $\frac{\sqrt{3}}{\sqrt{7}} = \left[\frac{\sqrt{3}}{\sqrt{7}}\right]\left[\frac{\sqrt{3}}{\sqrt{3}}\right] = \frac{3}{\sqrt{21}}$ ■

Problem: Rationalize the numerator in $\frac{7-\sqrt{3}}{\sqrt{5}}$.

$$(a-b)(a+b) = a^2 - b^2$$

Solution: $\frac{7-\sqrt{3}}{\sqrt{5}} = \left[\frac{7-\sqrt{3}}{\sqrt{5}}\right]\left[\frac{7+\sqrt{3}}{7+\sqrt{3}}\right]$

$$= \frac{7^2 - (\sqrt{3})^2}{\sqrt{5}(7+\sqrt{3})} = \frac{46}{\sqrt{5}(7+\sqrt{3})}$$ ■

Rationalize numerator and denominator

SKILL rationalize num&den

Problem: Rationalize the denomenator in $\dfrac{5}{\sqrt{x}}$.

Solution: $\dfrac{5}{\sqrt{x}} = \left[\dfrac{5}{\sqrt{x}}\right]\left[\dfrac{\sqrt{x}}{\sqrt{x}}\right] = \dfrac{5\sqrt{x}}{x}$ ■

Problem: Rationalize the numerator in $\dfrac{7-\sqrt{h}}{h}$.

Solution: $$\frac{7-\sqrt{h}}{h} = \left[\frac{7-\sqrt{h}}{h}\right]\left[\frac{7+\sqrt{h}}{7+\sqrt{h}}\right]$$
$$= \frac{49-h}{h(7+\sqrt{h})}$$ ■

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Rationalize numerator and denominator

SKILL rationalize num&den

Problem: Rationalize the numerator in $\dfrac{\sqrt{9+h}-3}{h}$.

Solution: $$\frac{\sqrt{9+h}-3}{h} = \left[\frac{\sqrt{9+h}-3}{h}\right]\left[\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right]$$

$$= \frac{(\cancel{9}+\overset{1}{\cancel{h}})-\cancel{9}}{\cancel{h}(\sqrt{9+h}+3)}$$

$$\overset{h\neq 0}{=} \frac{1}{\sqrt{9+h}+3} \blacksquare$$

STOP

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