CALCULUS
Absolute value and distance

absolute value **of** x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

absolute value of x

Definition: $\underset{5}{\boxed{|x|}} := \begin{cases} \overset{5}{x}, & \text{if } \overset{5}{x} \geq 0 \\ -x, & \text{if } \underset{5}{x} \leq 0 \end{cases}$

e.g.: $|5| = 5$ ■

absolute value of x

Definition: $$\underset{-5}{\boxed{|x|}} := \begin{cases} x, & \text{if } \overset{-5}{x} \geq 0 \\ \underset{-}{-}\underset{(-5)}{x}, & \text{if } \underset{-5}{x} \leq 0 \longleftarrow \end{cases}$$

$$-(-5)$$

e.g.: $|-5| = -(-5) = 5 \blacksquare$

SKILL
comp abs val

absolute value of x

Definition: $$\underset{0}{\boxed{|x|}} := \begin{cases} \overset{0}{x}, & \text{if } \overset{0}{x} \geq 0 \\ -x, & \text{if } \underset{0}{x} \leq 0 \end{cases}$$

$- \; 0$

e.g.: $|0| = 0$ ■

SKILL
comp abs val

$x :\rightarrow -3$

$\sqrt{x^2} = x$? NO

$\sqrt{(-3)^2} = -3$? NO

Fact: $\sqrt{x^2} = |x|$

Spp

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

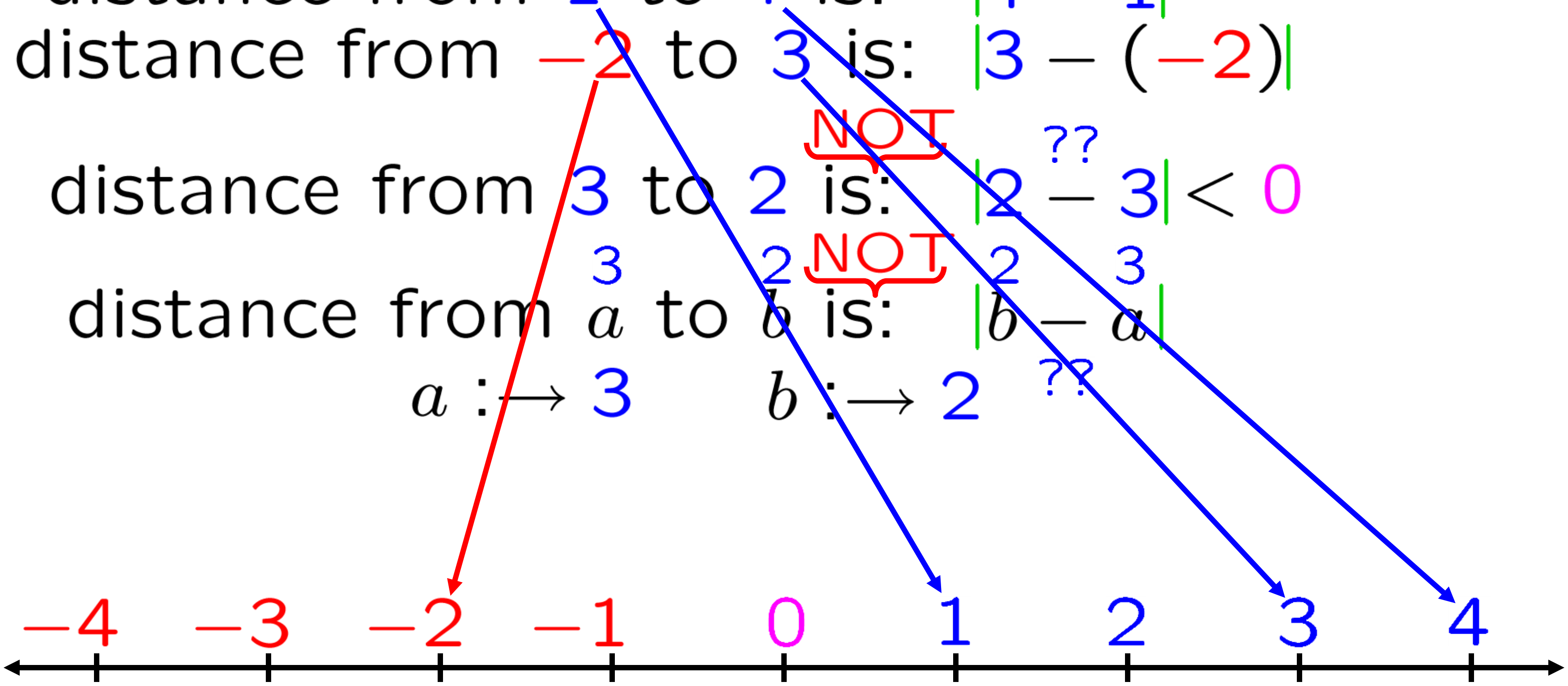
distance from 1 to 4 is: $|4-1|$

distance from -2 to 3 is: $|3-(-2)|$

distance from 3 to 2 $\overset{\text{NOT}}{\overbrace{\text{is:}}}$ $|2 \overset{??}{-} 3| < 0$

distance from $\overset{3}{a}$ to $\overset{2}{b}$ $\overset{\text{NOT}}{\overbrace{\text{is:}}}$ $|\overset{2}{b} \underset{??}{-} \overset{3}{a}|$

$a :\longrightarrow 3$ $\quad$ $b :\longrightarrow 2$



SKILL
comp dist

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x \le 0 \end{cases}$

$$|x| = |x - 0| = \text{dist}(x, 0)$$

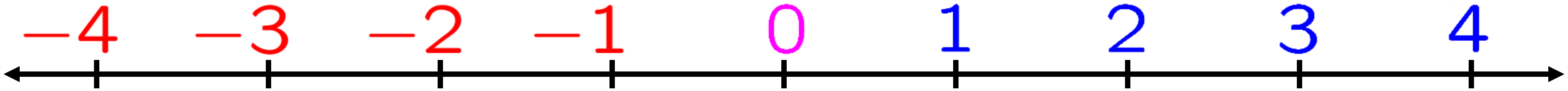
$$|x| \le r \quad \Leftrightarrow \quad -r \le x \le r$$

$$|x - a| = \text{dist}(x, a)$$

$$|x - a| \le r \quad \Leftrightarrow \quad a - r \le x \le a + r$$

distance from a to b is: $|b - a|$

$$\begin{array}{c} \| \\ .. \\ \boxed{\text{dist}(a, b)} \end{array}$$



absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

strict ineq.: $|x - a| \boxed{<} r \quad \Leftrightarrow \quad a - r \boxed{<} x \boxed{<} a + r$

$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

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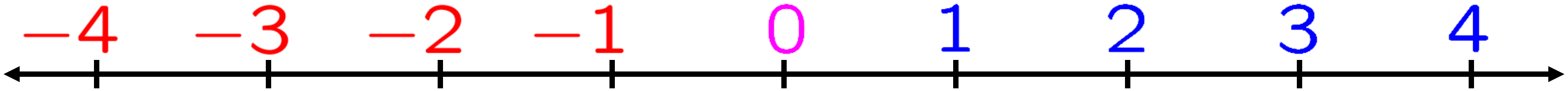
$\cdot\,\cdot$

$\boxed{\text{dist}(a, b)}$



absolute value of x

Definition: $$\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

$$|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$

$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

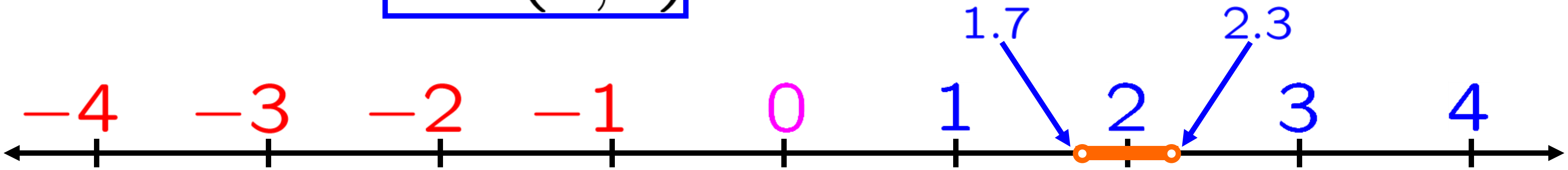
distance from a to b is: $|b - a|$

$\|$

$\ddot{}$

$\boxed{\text{dist}(a, b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x - 2| < 0.3$. $\Leftrightarrow$ $x \in (1.7, 2.3)$

absolute value of x

Definition: $$\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

$$|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$

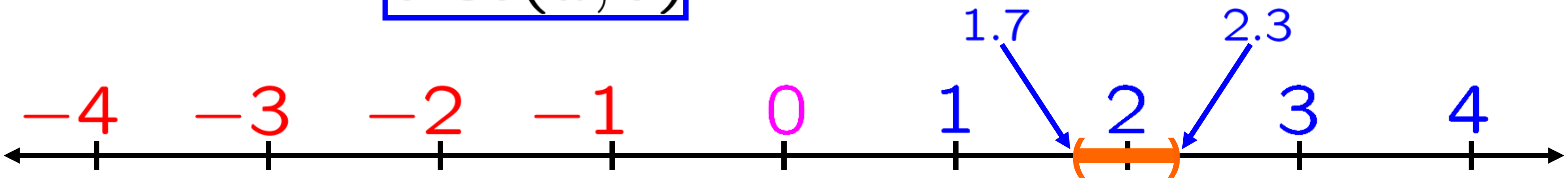
$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

distance from a to b is: $|b - a|$

$$\rotatebox{-90}{$=:$}$$

$$\boxed{\text{dist}(a, b)}$$

SKILL
gph abs val
inequality



Exercise: Graph $|x - 2| \enclose{circle}{<} 0.3. \Leftrightarrow x \in (1.7, 2.3)$

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absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

$$|x-a| < r \quad \Leftrightarrow \quad a-r < x < a+r$$

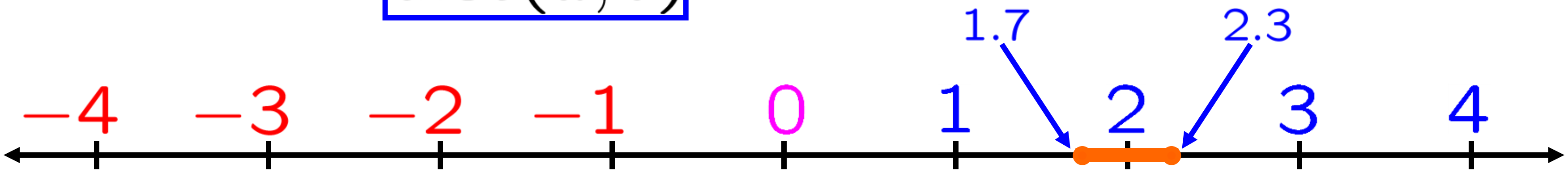
$$|x-a| \leq r \quad \Leftrightarrow \quad a-r \leq x \leq a+r$$

distance from a to b is: $|b-a|$

$\underset{..}{\|}$

$\boxed{\text{dist}(a,b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x-2| \textcircled{$\leq$} 0.3.$ $\Leftrightarrow$ $x \in [\underset{\uparrow}{1}.7, 2.3\underset{\uparrow}{]}$

Spp

absolute value of x

Definition: $\boxed{|x|} := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

$$|x - a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$

$$|x - a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

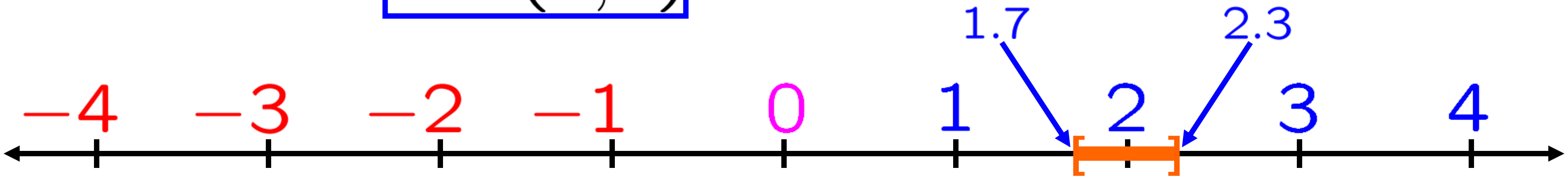
distance from a to b is: $|b - a|$

$\|$

$\cdot\cdot$

$\boxed{\text{dist}(a, b)}$

SKILL
gph abs val
inequality



Exercise: Graph $|x - 2| \leq 0.3$. $\Leftrightarrow$ $x \in [1.7, 2.3]$

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Exercises:

Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

$$|x-a| < r \quad \Leftrightarrow \quad a - r < x < a + r$$

$$|x-a| \leq r \quad \Leftrightarrow \quad a - r \leq x \leq a + r$$

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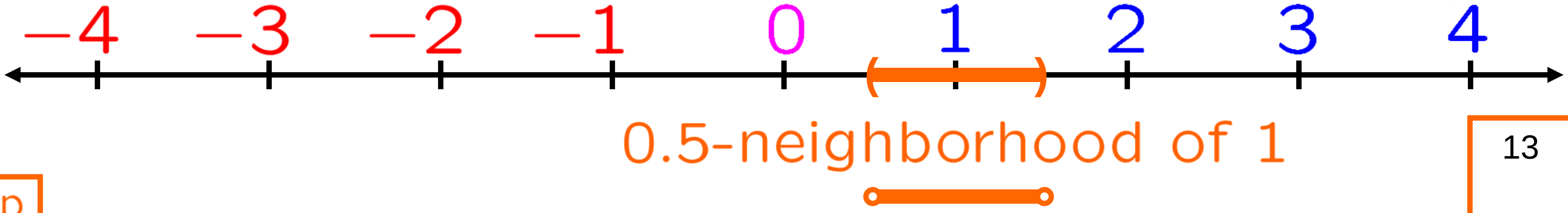
gph nbd



0.5-neighborhood of 1

Exercises:

Graph $|x-1|<0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0<|x-1|<0.5$. $\Leftrightarrow$ $x \in (0.5, 1.5)\boxed{\setminus}\{1\}$

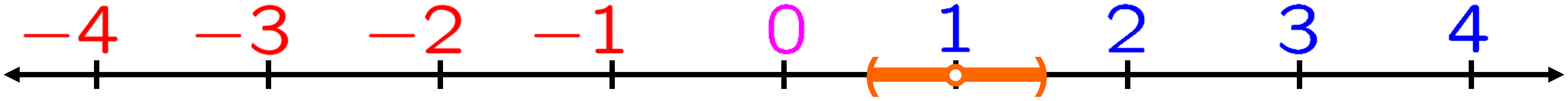
set-theoretic subtraction

$$|x-a|<r \quad \Leftrightarrow \quad a-r<x<a+r$$

$$|x-a|\leq r \quad \Leftrightarrow \quad a-r\leq x\leq a+r$$

SKILL
gph punctured nbd

SKILL
gph nbd



punctured 0.5-neighborhood of 1

Exercises:

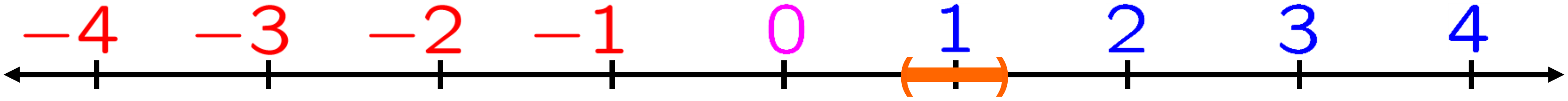
Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)$

SKILL
gph punctured nbd

SKILL
gph nbd



0.3-neighborhood of 1

Exercises:

Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)$

Graph $0 < |x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)\backslash\{1\}$

SKILL
gph punctured nbd

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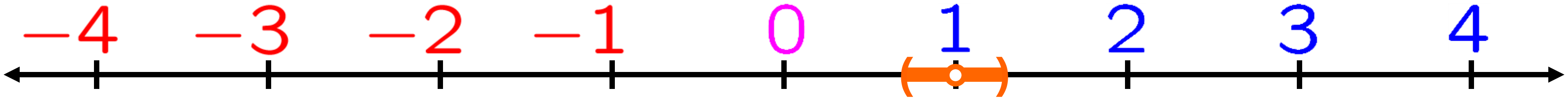

punctured 0.3-neighborhood of 1

Exercises:

Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)$

Graph $0 < |x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)\backslash\{1\}$

Graph $|x-1| < 0.1.$ $\Leftrightarrow$ $x \in (0.9, 1.1)$

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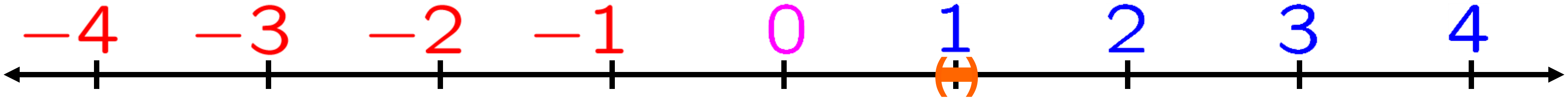

0.1-neighborhood of 1

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Exercises:

Graph $|x-1|<0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0<|x-1|<0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x-1|<0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)$

Graph $0<|x-1|<0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)\backslash\{1\}$

Graph $|x-1|<0.1.$ $\Leftrightarrow$ $x \in (0.9, 1.1)$

Graph $0<|x-1|<0.1.$ $\Leftrightarrow$ $x \in (0.9, 1.1)\backslash\{1\}$

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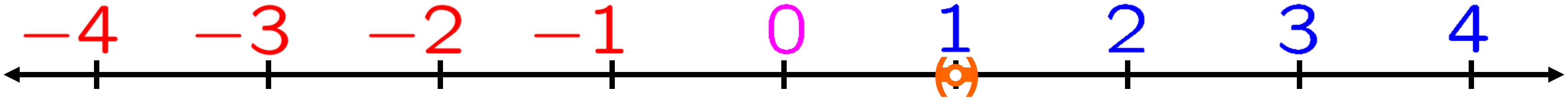
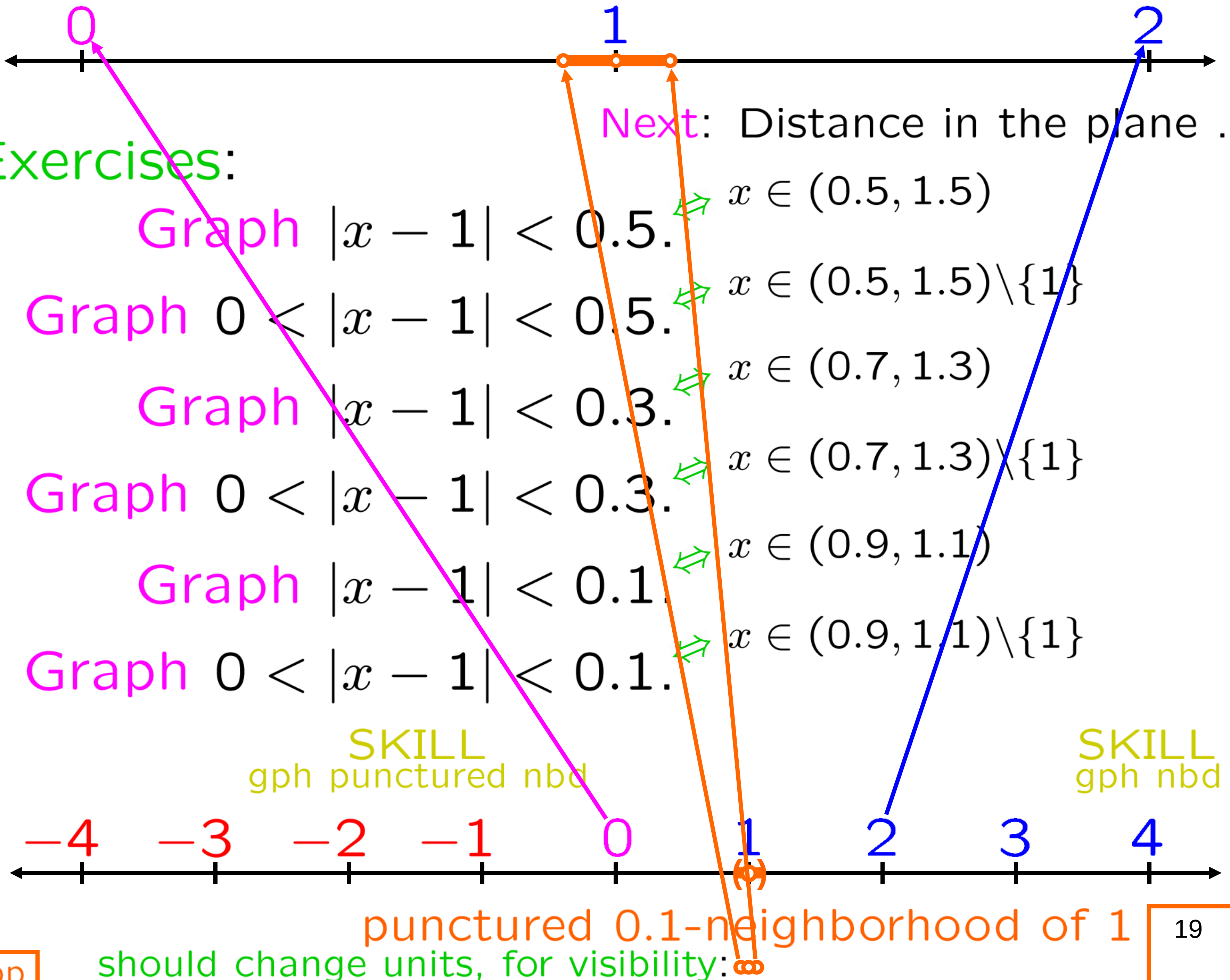
gph nbd



punctured 0.1-neighborhood of 1



Next: Distance in the plane . . .

Exercises:

Graph $|x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)$

Graph $0 < |x-1| < 0.5.$ $\Leftrightarrow$ $x \in (0.5, 1.5)\backslash\{1\}$

Graph $|x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)$

Graph $0 < |x-1| < 0.3.$ $\Leftrightarrow$ $x \in (0.7, 1.3)\backslash\{1\}$

Graph $|x-1| < 0.1.$ $\Leftrightarrow$ $x \in (0.9, 1.1)$

Graph $0 < |x-1| < 0.1.$ $\Leftrightarrow$ $x \in (0.9, 1.1)\backslash\{1\}$

SKILL
gph punctured nbd

SKILL
gph nbd

punctured 0.1-neighborhood of 1

should change units, for visibility:

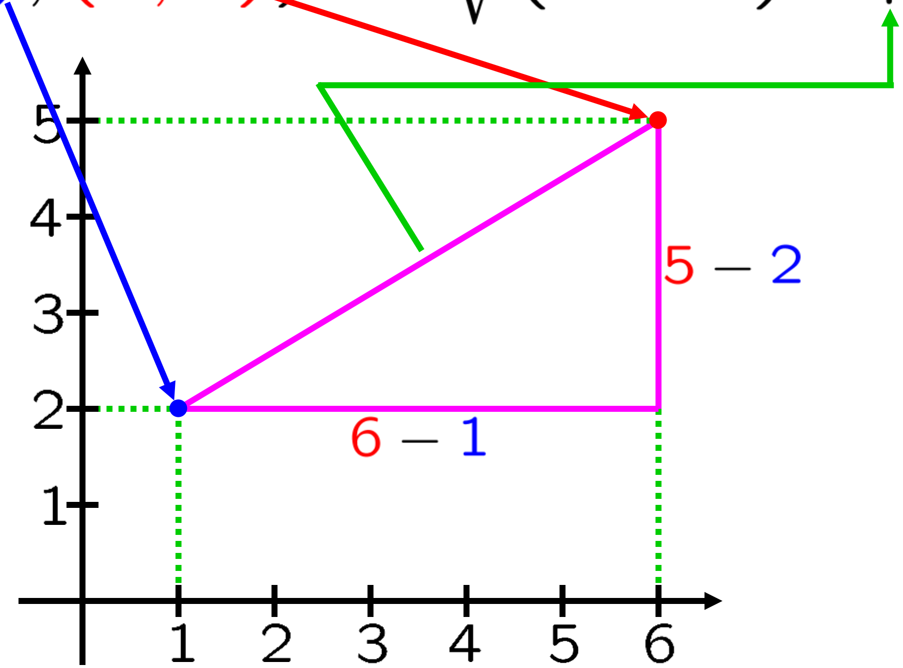
Spp

On the line,

$$\mathrm{dist}(a, s) = |s - a| = \sqrt{(s - a)^2}$$

In the plane,

$$\mathrm{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$



On the line,

$$\text{dist}(a, s) = |s - a| = \sqrt{(s-a)^2}$$

In the plane,

$$\text{dist}((1,2),(6,5)) = \sqrt{(6-1)^2 + (5-2)^2}$$

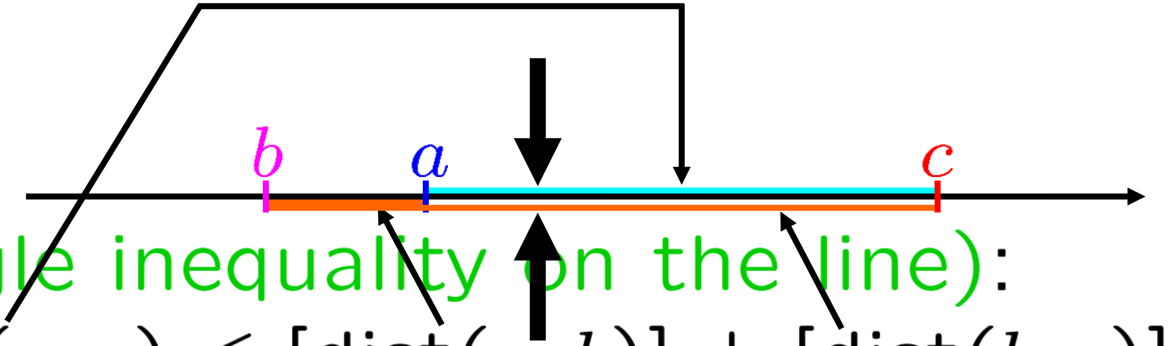
$$\text{dist}((a,b),(s,t)) = \sqrt{(s-a)^2 + (t-b)^2}$$

In the three dimensions,

SKILL
compute distances

$$\text{dist}((a,b,c),(s,t,u))$$

$$= \sqrt{(s-a)^2 + (t-b)^2 + (u-c)^2}$$



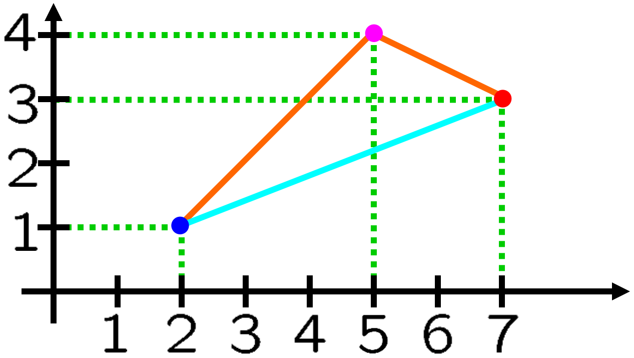
Fact (triangle inequality on the line):

$$\text{dist}(a,c) \leq [\text{dist}(a,b)] + [\text{dist}(b,c)]$$

Fact (triangle inequality on the plane):

$$\text{dist}((a,p),(c,r)) \leq [\text{dist}((a,p),(b,q))] + [\text{dist}((b,q),(c,r))]$$

e.g.: $\text{dist}((2,1),(7,3)) \leq [\text{dist}((2,1),(5,4))] + [\text{dist}((5,4),(7,3))]$



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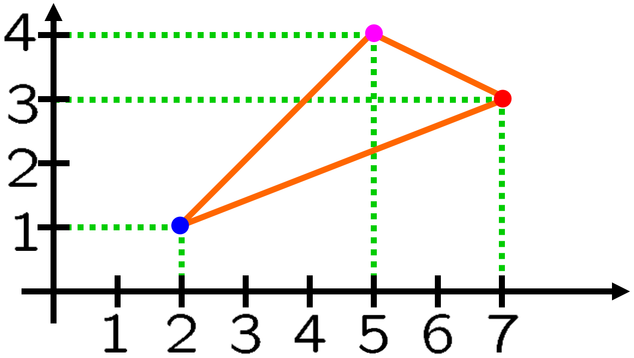
Fact (triangle inequality on the line):

$$\text{dist}(a,c) \le [\text{dist}(a,b)] + [\text{dist}(b,c)]$$

Fact (triangle inequality on the plane):

$$\text{dist}((a,p),(c,r)) \le [\text{dist}((a,p),(b,q))] + [\text{dist}((b,q),(c,r))]$$

e.g.: $\text{dist}((2,1),(7,3)) \le [\text{dist}((2,1),(5,4))] + [\text{dist}((5,4),(7,3))]$



Fact (triangle inequality on the line):

$$\underbrace{\mathsf{dist}(a,c)}_{|a-c|} \le [\underbrace{\mathsf{dist}(a,b)}_{|a-b|}] + [\underbrace{\mathsf{dist}(b,c)}_{|b-c|}]$$

Fact (triangle inequality on the line):

$$\begin{aligned} |a - c| \quad &\leq \quad \underbrace{|a - b|}_{} \quad + \quad \underbrace{|b - c|}_{} \\ |a - c| \quad &\leq \quad |a \; u \; b| \quad + \quad |b \; v \; c| \end{aligned}$$

Fact (triangle inequality on the line):

$$|\underbrace{a-c}_{}| \le |\underbrace{a-b}_{}| + |\underbrace{b-c}_{}|$$

$$|u+v| \le |u| + |v|$$

$$a - c = (\overbrace{a-\cancel{b}}^{u}) + (\overbrace{\cancel{b}-c}^{v})$$

“Absolute value is ‘subadditive’.”

Fact (triangle inequality on the line):

$$|u+v| \quad \leq \quad |u| \quad + \quad |v|$$

$$|u+v| \quad \leq \quad |u| \quad + \quad |v|$$

“The absolute value of the sum
is less than or equal to
the sum of the absolute values.”

“Absolute value is ‘subadditive’.”

Fact (triangle inequality on the line):

$$|u+v| \quad \leq \quad |u| \quad + \quad |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

Intuition:

$$\begin{aligned} x &\approx 3 && (\text{error} < 0.002) \\ y &\approx 5 && (\text{error} < 0.007) \\ x+y &\approx 3+5 && (\text{error} < ??) \end{aligned}$$

0.009

Pf. . .

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \le |u| + |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$\begin{aligned} x+y-3-5 &= (x-3)+(y-5) \\ \overbrace{|(x+y)-(3+5)|} &= |(x-3)+(y-5)| \\ &\le |x-3|+|y-5| \\ &< 0.002+0.007 \\ &= 0.009 \ \blacksquare \end{aligned}$$

“Absolute value is ‘subadditive’.”

Fact (triangle inequality on the line):

$$|u+v| \quad \le \quad |u| \quad + \quad |v|$$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$|(x+y)-(3+5)| < 0.002 + 0.007$$

$$|(x+y)-(3+5)|$$

Using algebra and the triangle inequality,
we have proved this from these.

$$< 0.002 + 0.007$$

"Absolute value is 'subadditive'."

Fact (triangle inequality on the line):

$$|u+v| \quad \le \quad |u| \quad + \quad |v|$$

$3 :\rightarrow s$ $\quad 0.002 :\rightarrow \sigma$ $\quad 5 :\rightarrow t$ $\quad 0.007 :\rightarrow \tau$

Suppose $|x-3| < 0.002$ and $|y-5| < 0.007$.

How close is $x+y$ to $3+5$?

$$|(x+y)-(3+5)| < 0.002 + 0.007$$

Fact (additivity of error):

$$|x-s| < \sigma \text{ and } |y-t| < \tau$$

$$\Rightarrow \quad |(x+y)-(s+t)| < \sigma + \tau$$

Exercise: Using algebra and the triangle inquality, prove this from these.